Prime and Maximal Ideals in Commutative Rings

Vocabulary: prime, maximal, zero divisor

A ***ring*** which can be written in the form R/I is called a ***quotient ring*** of R.

Definition. Suppose R is a commutative[a] ring. An ideal P of R is called ***prime*** if,

- for all a and $b \in R$, if $ab \in P$ then $a \in P$ or $b \in P$.
- The ideal P is not[b] all of R.

[a]We will discuss prime and maximal ideals in non-commutative rings later.

[b]This second condition may seem ***ad hoc***; it is a good idea for the same reason that we should define 1 **not** to be a prime number.

(26) An element a of a commutative ring S is called a ***zero divisor*** if there is some $x \neq 0$ in S for which $ax = 0$. Let R be a commutative ring and let I be an ideal of R. Show that I is prime if and only if R/I has no nonzero zero divisors, but does have some nonzero element.

(27) What are the prime ideals in $\mathbb{Z}$? You may assume uniqueness of prime factorization for this question. [1]

Definition. Suppose R is a commutative ring. An ideal $\mathfrak{m}$ of R is called ***maximal*** if,

- for all a in R, if $a \notin \mathfrak{m}$ then there is some $b \in R$ such that $ab \equiv 1 \mod \mathfrak{m}$.
- The ideal $\mathfrak{m}$ is not all of R.

(28) Let R be a commutative ring and let I be an ideal of R. Show that I is maximal and only if R/I is a field.

(29) Show that a maximal ideal is prime.

(30) Show that an ideal $I \subsetneq R$ is maximal if and only there does not exist an ideal J with $I \subsetneq J \subsetneq R$.

Problem (30) is the motivation for the word "maximal". Using Zorn's lemma, and Problem (30), it is easy to show that every ideal in a nonzero commutative ring is contained in a maximal ideal.

(31) Let $R = \mathbb{R}[x, y]$. Show that yR is prime but not maximal.

(32) What are the maximal ideals of $\mathbb{Z}$?

[1]In a week or so, we will discuss unique factorization in commutative rings in general. At that point, we will prove it for $\mathbb{Z}$. The careful student can check that there is no circularity; the problems where I permit you to use it now will not feed into our proof then.