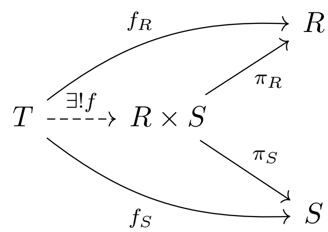
# PRODUCTS FROM A CATEGORICAL PERSPECTIVE

The product operation is ubiquitous in mathematics, What do these notions of product have in common? Universal properties are a systematic tool for approaching this question.

As is often true of category theory, it is both true that one *can* get a very long way without explicitly addressing these issues, and that everything becomes clearer if you do.

**Universal Property of Products in Rings**. Suppose $R$ and $S$ are rings. A ***categorical product*** of $R$ and $S$ is a ring, denoted $R \times S$, together with two ring homomorphisms $\pi_R \colon R \times S \to R$ and $\pi_S \colon R \times S \to S$ such that for every ring $T$ and every pair of ring homomorphisms $f_R \colon T \to R$ and $f_S \colon T \to S$ there is a unique ring homomorphism $f \colon T \to R \times S$ for which the following diagram commutes.

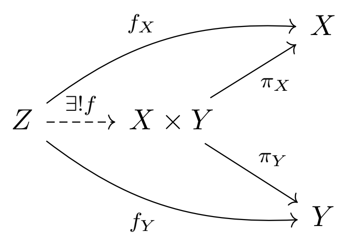



The maps $\pi_R$ and $\pi_S$ are usually called projections and $f$ is often denoted as $f_R \times f_S$.

(87) Show that the product ring $R \times S$, with its natural projection maps, is a categorical product in the sense above.

For a general category, the universal property of products looks like:

**Universal Property of products**. Suppose $\mathcal{C}$ is a category. For objects $X$ and $Y$ in $\mathcal{C}$, a product of $X$ and $Y$ is an object in $\mathcal{C}$, denoted $X \times Y$, together with a pair of morphisms $\pi_X \colon X \times Y \to X$ and $\pi_Y \colon X \times Y \to Y$ that satisfy the following universal property: for every object $Z$ in $\mathcal{C}$ and pair of morphisms $f_X \colon Z \to X$, $f_Y \colon Z \to Y$ there exists a unique morphism $f \colon Z \to X \times Y$ such that the following diagram commutes:

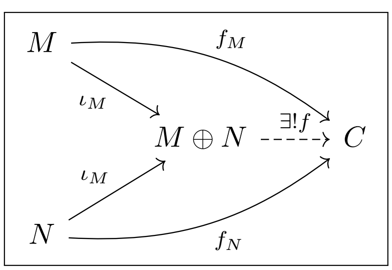



Warning: products need not exist! However, when they exist, they are unique up to unique isomorphism

(88) Suppose $Q$ is an object in $\mathcal{C}$ that, along with morphisms $\tilde{\pi}_X \colon Q \to X$ and $\tilde{\pi}_Y \colon Q \to Y$, satisfies the universal property of products. Show that there exists a natural isomorphism[1] from $Q$ to $X \times Y$.

(89) Let $R$ be a ring and let $M$ and $N$ be $R$-modules. Show that $M \oplus N$ is the product of $M$ and $N$ in the category of $R$-modules.

(90) Let $R$ be a ring and let $M$ and $N$ be $R$-modules. Let $\iota_M : M \to M \oplus N$ and $\iota_N : N \to M \oplus N$ be the maps $m \mapsto (m, 0)$ and $n \mapsto (0, n)$. Show that $M \oplus N$ is also the coproduct of $M$ and $N$ in the category of $R$-modules, meaning that, given any $R$-module $C$ and any $R$-module maps $f_M : M \to C$ and $f_N : N \to C$, there is a unique $R$-module map $f : M \oplus N \to C$ such that the following diagram commutes:




(91) Is $R \times S$ the coproduct of $R$ and $S$ in the category of rings? What if we allow ring maps to be non-unital?

[1]For $A, B$ in $\mathcal{C}$ we say $T \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is an ***isomorphism*** provided there exists $S \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ for which $S \circ T = \mathrm{Id}_A$ and $T \circ S = \mathrm{Id}_B$.