Rational canonical form of a matrix

Vocabulary: Companion matrix, rational canonical form

Let k be a field and let $f = x^d + f_{d-1}x^{d-1} + \cdots + f_0$ be a monic polynomial with coefficients in k. We define the ***companion matrix*** of f by

$$\mathcal{C}(f) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -f_0 \\ 1 & 0 & 0 & \cdots & 0 & -f_1 \\ 0 & 1 & 0 & \cdots & 0 & -f_2 \\ 0 & 0 & 1 & \cdots & 0 & -f_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -f_{d-1} \end{bmatrix}$$

(110) Show that $k[x]/f(x)k[x]$ is a k-vector space of dimension d, and we can choose a basis of this vector space in which multiplication by x is given by the matrix $\mathcal{C}(f)$.

(111) Compute the trace and determinant of multiplication by x, as a linear map $k[x]/f(x)k[x] \longrightarrow k[x]/f(x)k[x]$.

An $n \times n$ matrix with entries in k is said to be in ***rational***[1] ***canonical form*** (also known as Frobenius normal form) if it is a block matrix of the form

$$\begin{bmatrix} \boxed{\mathcal{C}(f_1)} & & & \\ & \boxed{\mathcal{C}(f_2)} & & \\ & & \ddots & \\ & & & \boxed{\mathcal{C}(f_k)} \end{bmatrix}$$

for some monic polynomials $f_1(x), f_2(x), \ldots, f_k(x)$ with $f_1 | f_2 | \cdots | f_k$.

(112) Let V be a finite dimensional k-vector space and let $T : V \to V$ be a k-linear map. Show that there is a basis of V in which T is given by a matrix in rational canonical form, and that the polynomials $f_1, f_2, \ldots, f_k$ are uniquely determined by (V, T).

(113) Let X be an $n \times n$ matrix with entries in k. Show that there is an invertible matrix S such that SXS^{-1} is in rational canonical form, and that the polynomials $f_1, f_2, \ldots, f_k$ are uniquely determined by X.

We recall that $n \times n$ matrices X and Y are called ***similar*** if there is an invertible matrix S with $Y = SXS^{-1}$.

(114) Describe the characteristic polynomial of T in terms of $f_1, f_2, \ldots, f_k$.

(115) The ***minimal polynomial*** of T is the monic polynomial $g(t) \in k[t]$ of lowest degree such that $g(T) = 0$. Describe the minimal polynomial of T in terms of $f_1, f_2, \ldots, f_k$.

(116) Let $K \subset L$ be fields and let X be an $n \times n$ matrix with entries in K. Show that X has the same rational canonical form when considered as a matrix over K or over L.

[1] The word "rational" is because we can put matrices into rational canonical form while staying in the same ground field, unlike Jordan-canonical form where need to pass to a larger field. It does not indicate that the notion is special to the field $\mathbb{Q}$.