Introduction to Smith normal form

The Smith[1] normal form theorem says the following:

Theorem (Smith normal form). Let R be a principal ideal domain and let X be an $m \times n$ matrix with entries in R. Then there invertible $m \times m$ and $n \times n$ matrices U and V, and elements $d_1, d_2, \ldots, d_{\min(m,n)}$ of R, such that

$$X = UDV,$$

where D is the $m \times n$ matrix with $D_{jj} = d_j$ and $D_{ij} = 0$ for $i \neq j$. Moreover, we may assume $d_1 | d_2 | \cdots | d_{\min(m,n)}$ and, with this normalization, the d_j are unique up to multiplication by units.

The d_j are called the ***invariant factors*** of X. We first set up some notation:

(92) Let R be any ring. Define an relation $\sim$ on $\mathrm{Mat}_{m\times n}(R)$ by $X \sim Y$ if there are invertible $m \times m$ and $n \times n$ matrices U and V with $Y = UXV$. Show that $\sim$ is an equivalence relation.[2]

(93) Here is a more abstract perspective on $\sim$: Let X and $Y \in \mathrm{Mat}_{m\times n}(R)$.

(a) Show that $X \sim Y$ if and only if we can choose vertical isomorphisms making the following diagram commute:

$$\begin{tikzcd} R^n \arrow[r, "X"] \arrow[d, "\cong"] & R^m \arrow[d, "\cong"] \\ R^n \arrow[r, "Y"] & R^m \end{tikzcd}$$

(b) Show that, if $X \sim Y$, then the kernels, cokernels and images of X and Y are isomorphic R-modules.[3]

For nonnegative integers m and n and elements $d_1, d_2, \ldots, d_{\min(m,n)}$ of R, we define $\mathrm{diag}_{mn}(d_1, d_2, \ldots, d_{\min(m,n)})$ to be the $m \times n$ matrix D above. Thus, Smith normal form says that every matrix is $\sim$-equivalent to a matrix of the form $\mathrm{diag}_{mn}(d_1, d_2, \ldots, d_{\min(m,n)})$ with $d_1|d_2|\cdots|d_{\min(m,n)}$ and the d_j are unique up to multiplication by units.

It will be convenient today to know the following formula. The morally right proof of this result will be more natural in a month[4], so you may assume it for now.

The Cauchy-Binet formula. Let R be a commutative ring. Given an $m \times n$ matrix X with entries in R, and subsets $I \subseteq \{1, 2, \ldots, m\}$ and $J \subseteq \{1, 2, \ldots, n\}$ of the same size, define $\Delta_{IJ}(X)$ to be the determinant of the square submatrix of X using rows I and columns J. Let X and Y be $a \times b$ and $b \times c$ matrices with entries in R and let I and K be subsets of $\{1, 2, \ldots, a\}$ and $\{1, 2, \ldots, c\}$ with $|I| = |J| = q$. Then

$$\Delta_{IK}(XY) = \sum_{J \subseteq \{1,2,\ldots,b\},\ |J|=q} \Delta_{IJ}(X)\Delta_{JK}(Y).$$

The next few problems show how to compute invariant factors.

(94) Let R be a UFD. Let U, X and V be $m \times m$, $m \times n$ and $n \times n$ matrices with entries in R. Show that the GCD of the $q \times q$ minors of X divides the GCD of the $q \times q$ minors of UXV.

(95) Let R be a UFD. Show that, if $X \sim Y$, then the GCD of the $q \times q$ minors of X is equal to the GCD of the $q \times q$ minors of Y.

(96) Let R be a UFD. Let X be an $m \times n$ matrix with entries in R. Show that, if $X \sim \mathrm{diag}_{mn}(d_1, d_2, \ldots, d_{\min(m,n)})$ with $d_1|d_2|\cdots|d_{\min(m,n)}$, then $d_1 d_2 \cdots d_q$ is the GCD of the $q \times q$ minors of X. Deduce that invariant factors are uniquely defined up to multiplication by units.

(97) Assuming the Smith normal form theorem for $\mathbb{Z}$, compute the invariant factors of the following matrices:

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

(98) If you have gotten this far, go ahead and prove the Cauchy-Binet formula. It can be done by brute force.

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[1] Named for Henry John Stephen Smith, an Irish mathematician who lived from 1826 to 1883.

[2] The factorization UDV may remind the reader of singular value decomposition. This is not a coincidence; Smith normal form can be thought of as a non-Archimedean version of singular value decomposition.

[3] The converse does not hold, see `https://mathoverflow.net/questions/343143`.

[4] The proof appeared as problem 182. This worksheet was done on October 16 and that one was on November 25, so one month was a slight underestimate.