TENSOR ALGEBRAS, SYMMETRIC AND EXTERIOR ALGEBRAS

For this worksheet, we move back to the world of vector spaces. It is possible to study these concepts over a general commutative ring, but this seems like enough for now.

Let v be a field and let V be a vector space over k. By problem 157, there is a natural isomorphism $(V \otimes V) \otimes V \cong V \otimes (V \otimes V)$ and similarly for higher tensor powers. We therefore write $V^{\otimes n}$ for the n-fold tensor product of V with itself and write elements of $V^{\otimes n}$ as $\sum c_{j_1 j_2 \cdots j_n} \, v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_n}$ without parentheses. We define $V^{\otimes 0}$ to be k.

We define the ***tensor algebra*** $T(V)$ to be $\bigoplus_d V^{\otimes d}$.

(169) Show that $T(V)$ has a unique ring structure where the product of $\sigma \in V^{\otimes s}$ and $\tau \in V^{\otimes t}$ is $\sigma \otimes \tau \in V^{\otimes (s+t)}$.

(170) Let $\alpha : V \to W$ be a linear map. Show that there is a unique map of rings $T(\alpha) : T(V) \to T(W)$ with $T(\alpha)(v) = \alpha(v)$ for $v \in V$.

We define the symmetric algebra $\mathrm{Sym}^\bullet(V)$ to be the quotient of $T(V)$ by the 2-sided ideal generated by all tensors of the form $v \otimes w - w \otimes v$.

(171) Show that $\mathrm{Sym}^\bullet(V)$ is a commutative ring.

(172) Show that $\mathrm{Sym}^\bullet(V)$ breaks up as a direct sum $\bigoplus_{d=0}^{\infty} \mathrm{Sym}^d(V)$ where $\mathrm{Sym}^d(V)$ is a quotient of $V^{\otimes d}$.

(173) Let $x_1, x_2, \ldots, x_n$ be a basis of V. Show that $\{x_{i_1} x_{i_2} \cdots x_{i_d} \ : \ 1 \leq i_1 \leq i_2 \leq \cdots \leq i_d \leq n\}$ is a basis of $\mathrm{Sym}^d(V)$. Show that $\mathrm{Sym}^\bullet(V) \cong k[x_1, \ldots, x_n]$.

We define the exterior algebra, $\bigwedge^\bullet(V)$ to be the quotient of $T(V)$ by the two sided ideal generated by $v \otimes v$ for all $v \in V$. The multiplication in $\bigwedge^\bullet(V)$ is generally denoted $\wedge$.

(174) Show that, for v and $w \in V$, we have $v \wedge w = -w \wedge v$.

(175) Show that $\bigwedge^\bullet(V)$ breaks up as a direct sum $\bigoplus_{d=0}^{\infty} \bigwedge^d(V)$ where $\bigwedge^d(V)$ is a quotient of $V^{\otimes d}$.

(176) Let $e_1, e_2, \ldots, e_n$ be a basis of V. Show that $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_d} \ : \ 1 \leq i_1 < i_2 < \cdots < i_d \leq n\}$ is a basis of $\bigwedge^d(V)$.

(177) Let $v_1, v_2, \ldots, v_d \in V$. Show that $v_1 \wedge v_2 \wedge \cdots \wedge v_d = 0$ if and only if $v_1, v_2, \ldots, v_d$ are linearly dependent.

We now consider the effect of these constructions on linear maps. Let V and W be k-vector spaces and $\alpha : V \to W$ a linear map.

(178) Show that there are unique ring maps $\mathrm{Sym}^\bullet(\alpha) : \mathrm{Sym}^\bullet(V) \to \mathrm{Sym}^\bullet(W)$ and $\bigwedge^\bullet(\alpha) : \bigwedge^\bullet(V) \to \bigwedge^\bullet(W)$ with $\mathrm{Sym}^\bullet(\alpha)(v) = \alpha(v)$ and $\bigwedge^\bullet(\alpha)(v) = \alpha(v)$ for $v \in V$.

(179) Let $\alpha : k^3 \to k^3$ be given by the matrix $\left[\begin{smallmatrix} r & s & t \\ u & v & w \\ x & y & z \end{smallmatrix}\right]$. Compute the matrix of $\bigwedge^2(\alpha) : \bigwedge^2(k^3) \to \bigwedge^2(k^3)$.

(180) Let $\alpha : k^2 \to k^2$ be given by the matrix $\left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right]$. Compute the matrix of $\mathrm{Sym}^2(\alpha) : \mathrm{Sym}^2(k^2) \to \mathrm{Sym}^2(k^2)$.

(181) Show $\bigwedge^d(\alpha \circ \beta) = \bigwedge^d(\alpha) \circ \bigwedge^d(\beta)$ and $\mathrm{Sym}^d(\alpha \circ \beta) = \mathrm{Sym}^d(\alpha) \circ \mathrm{Sym}^d(\beta)$.

Given an $m \times n$ matrix X with entries in k, and subsets $I \subseteq \{1, 2, \ldots, m\}$ and $J \subseteq \{1, 2, \ldots, n\}$ of the same size, define $\Delta_{IJ}(X)$ to be the determinant of the square submatrix of X using rows I and columns J.

(182) Prove the Cauchy-Binet formula: Let X and Y be $a \times b$ and $b \times c$ matrices with entries in k and let I and K be subsets of $\{1, 2, \ldots, a\}$ and $\{1, 2, \ldots, c\}$ with $|I| = |J| = q$. Then

$$\Delta_{IK}(XY) = \sum_{\substack{J \subseteq \{1, 2, \ldots, b\} \\ |J| = q}} \Delta_{IJ}(X) \Delta_{JK}(Y).$$