Tensor products of modules

Let R be a ring, let M be a right R-module and N a left R-module. We define the abelian group $M \otimes N$ to be generated by symbols $m \otimes n$, for $m \in M$ and $n \in N$, modulo the relations:

$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2, \ (m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n, \ (mr) \otimes n = m \otimes (rn).$$

We write $M \otimes_R N$ if the ring R is not clear from context.

(145) Prove the ***universal property of tensor products***: For any abelian group A, and any R-bilinear pairing $\langle\ ,\ \rangle : M \times N \to A$, there is a unique additive map $\lambda : M \otimes N \to A$ such that $\langle m, n \rangle = \lambda(m \otimes n)$. (In the noncommutative setting, R-bilinear just means $\langle mr, n \rangle = \langle m, rn \rangle$.)

(146) Let M_1 and M_2 be right R-modules, N_1 and N_2 be left R-modules and $\alpha : M_1 \to M_2$ and $\beta : N_1 \to N_2$ be R-linear maps. Show that there is a unique additive map $\alpha \otimes \beta : M_1 \otimes N_1 \to M_2 \otimes N_2$ such that $(\alpha \otimes \beta)(m \otimes n) = \alpha(m) \otimes \beta(n)$.

(147) Let M_1, M_2, M_3 be right R-modules, let N_1, N_2 and N_3 be left R-modules, and let $\alpha_1 : M_1 \to M_2$, $\alpha_2 : M_2 \to M_3$, $\beta_1 : N_1 \to N_2$ and $\beta_2 : N_2 \to N_3$ be R-linear maps. Show that $(\alpha_2 \otimes \beta_2) \circ (\alpha_1 \otimes \beta_1) = (\alpha_2 \circ \alpha_1) \otimes (\beta_2 \otimes \beta_1)$.

Tensor products over noncommutative rings are important, but we will mostly focus on the commutative case.

From now on, let R be a commutative ring.

(148) Let M and N be R-modules. Show that there is a unique R-module structure on $M \otimes N$ such that $r(m \otimes n) = (rm) \otimes n = m \otimes (rn)$.

(149) Let M, N and A be R-modules and let $M \otimes N \to A$ be an R-bilinear pairing (which now means that $\langle rm, n \rangle = \langle m, rn \rangle = r\langle m, n \rangle$). Show that the unique map $\lambda : M \otimes N \to A$ coming from the universal property of tensor products is R-linear.

(150) Suppose that I is a subset of M which generates M as an R-module, and J is a subset of N which generates N as an R-module. Show that the elements $m \otimes n$, $m \in I$ and $n \in J$, generate $M \otimes N$ as an R-module.

(151) Show that $(M_1 \oplus M_2) \otimes_R N \cong M_1 \otimes_R N \oplus M_2 \otimes_R N$ and $M \otimes_R (N_1 \oplus N_2) \cong M \otimes_R N_1 \oplus M \otimes_R N_2$.

(152) Show that $R \otimes_R M \cong M$ and $M \otimes_R R \cong M$.

(153) Show that $R^{\oplus m} \otimes R^{\oplus n} \cong R^{\oplus mn}$.

So far we have emphasized the similarities between tensor products of vector spaces and of modules, but there are important differences.

(154) It is **not** true that, if $M_1 \to M_2$ is injective then $M_1 \otimes N \to M_2 \otimes N$ is injective. Give a counterexample. One of the easiest takes $R = \mathbb{Z}$ and $N = \mathbb{Z}/2\mathbb{Z}$.

(155) It is **not** true that, if $m \in M$ and $n \in N$ are nonzero, then $m \otimes n$ is nonzero in $M \otimes N$. Give a counterexample. One of the easiest takes $R = \mathbb{Z}$.