MORE TENSOR PRODUCT PROBLEMS

We recall: Let R be a ring, let M be a right R-module and N a left R-module. We define the abelian group $M \otimes N$ to be generated by symbols $m \otimes n$, for $m \in M$ and $n \in N$, modulo the relations:

$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2, \ (m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n, \ (mr) \otimes n = m \otimes (rn).$$

We write $M \otimes_R N$ if the ring R is not clear from context. If the ring R is commutative, then $M \otimes_R N$ is an R-module, with $r(m \otimes n) = (rm) \otimes n = m \otimes (rn)$.

Today's problems are a list of things I think you should know; it is less cohesive than usual.

(156) Let R be a commutative ring and let A and B be R-modules. Show that

$$A \otimes_R B \cong B \otimes_R A.$$

(157) Let R be a commutative ring and let A, B and C be R-modules. Show that

$$(A \otimes_R B) \otimes_R C \cong A \otimes_R (B \otimes_R C).$$

(158) Let R be a commutative ring and let I and J be ideals of R. The goal of this problem is to show that $R/I \otimes_R R/J \cong R/(I+J)$ as an R-module.

- (a) Show that the R-module $R/I \otimes_R R/J$ is generated as an R-module by $1 \otimes 1$. In other words, show that the map $r \mapsto r(1 \otimes 1)$ from R to $R/I \otimes_R R/J$ is surjective.
- (b) Show that the kernel of the map in the previous part is $I + J$.

(159) To check that you understood Problem (158), describe the $\mathbb{Z}$-module $(\mathbb{Z}/6\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/10\mathbb{Z})$ in as simple terms as possible.

Let R and S be two rings. An (R, S)***-bimodule*** is a set M with maps $+_M : M \times M \to M$, $\times_R : R \times M \to M$ and $\times_S : M \times S \to M$ such that $(+_M, \times_R)$ makes M into a left R-module, $(+_M, \times_S)$ makes M into a right S-module and, for $r \in R$, $m \in M$ and $s \in S$, we have $(r \times_R m) \times_S s = r \times_R (m \times_S s)$. This is the answer to "how do you put a module structure on a tensor product over a non-commutative ring?"

(160) Let R, S and T be rings. Let M be an (R, S)-bimodule and let N be an (S, T)-bimodule. Show that $M \otimes_S N$ has a unique (R, T)-bimodule structure for which $r \times_R (m \otimes n) = (r \times_R m) \otimes n$ and $(m \otimes n) \times_T t = m \otimes (n \times_T t)$ for $r \in R$, $m \in M$, $n \in N$ and $t \in T$.

For the next problem, we move back to the land of vector spaces:

(161) Let V and W be k-vector spaces, and let $V^\vee$ be the dual vector spaces.

- (a) Construct a "natural" linear map $\alpha : V^\vee \otimes_k W \to \mathrm{Hom}_k(V, W)$.
- (b) Show that the image of this map is the set of ϕ in $\mathrm{Hom}_k(V, W)$ for which $\dim \phi(V) < \infty$.
- (c) Show that, if V or W is finite dimensional, then α is surjective.
- (d) Show that, if V and W are finite dimensional, then α is an isomorphism.

The aim of the next several parts is to show that α is always injective (and thus, in particular, α is an isomorphism if either V or W is finite dimensional).

- (e) Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be linearly independent elements of $V^\vee$. Show that, for each index j, there is a vector $v \in V$ with $\lambda_j(v) = 1$ and $\lambda_i(v) = 0$ for $i \neq j$.
- (f) Suppose, for the sake of contradiction, that $\alpha\left(\sum \lambda_j \otimes w_j\right) = 0$. Show that $\sum \lambda_j \otimes w_j = 0$. Hint: First reduce to the case that the λ_j are linearly independent.