# TENSOR PRODUCTS OF RINGS

**Throughout this worksheet, let $k$ be a commutative ring.**

Recall that an associative unital $k$-algebra is a ring $R$ equipped with a map of rings from $k$ to the center of $R$. Note that, in particular, a $k$-algebra is a $k$-module.

**Convention for this worksheet: I will shorten "associative unital $k$-algebra" to "$k$-algebra".**

(162) Let $R$ and $S$ be $k$-algebras. Show that $R \otimes_k S$ has a unique structure of $k$-algebra such that $(r_1 \otimes s_1)(r_2 \otimes s_2) = (r_1 r_2) \otimes (s_1 s_2)$.

(163) Show that $k[x] \otimes_k k[y] \cong k[x, y]$.

(164) Let $G$ and $H$ be groups and let $kG$ and $kH$ be the corresponding group rings. Show that $kG \otimes_k kH \cong k(G \times H)$.

(165) Let $R$ be a commutative ring and let $I$ and $J$ be ideals of $R$. Show that $(R/I) \otimes_R (R/J) \cong R/(I + J)$ as rings. You already proved this isomorphism as $R$-modules back in Problem (158).

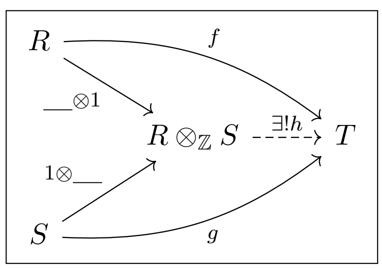
(166) Let $A = k[x_1, x_2]/(x_1^2 + x_2^2 + 1)$ and let $B = k[y_1, y_2, y_3]/(y_1 + y_2 + y_3)$. Show that

$$A \otimes_k B \cong k[x_1, x_2, y_1, y_2, y_3]/(x_1^2 + x_2^2 + 1,\ y_1 + y_2 + y_3).$$

Here we use parenthesis for the ideal in the relevant ring generated by the parenthesized elements.

(167) Generalize Problems (165) and (166) to a statement whose form is: "Let $R$ and $S$ be commutative $k$-algebras, and $I$ and $J$ ideals of $R$ and $S$ respectively. Then $(R/I) \otimes_k (S/J)$ is isomorphic to ...".

A month ago, one of you asked what the coproduct is in the category of rings. We are now ready to answer that for commutative rings: For $R$ and $S$ commutative rings, the coproduct of $R$ and $S$ is $R \otimes_{\mathbb{Z}} S$. This means that, given any commutative ring $T$ and any maps of rings $f : R \to T$ and $g : S \to T$, there is a unique map $h : R \otimes_{\mathbb{Z}} S \to T$ making the diagram below commute:




(168) Prove the above statement.