TENSOR PRODUCTS OF VECTOR SPACES

"*I wasn't asking much: I just wanted to figure out the most basic properties of tensor products. And it seemed like a moral issue. I felt strongly that if I really really wanted to feel like I understand this ring, which is after all a set, then at least I should be able to tell you, with moral authority, whether an element is zero or not. For fuck's sake!*"
"What tensor products taught me about living my life" (Cathy O'Neil),
`https://mathbabe.org/2011/07/20/what-tensor-products-taught-me-about-living-my-life/`

Let k be a field and let V and W be k-vector spaces. Define $V \otimes W$ to be the k-vector space generated by symbols $v \otimes w$, for $v \in V$ and $w \in W$, modulo the following relations:

$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2 \quad (v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w \quad c(v \otimes w) = (cv) \otimes w = v \otimes (cw) \quad (*).$$

Here $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$ and $c \in k$.

(135) Show that $0 \otimes w = v \otimes 0 = 0$.

(136) Prove the ***universal property of tensor products***: For any vector space k, and any k-bilinear pairing $\langle\ ,\ \rangle : V \times W \to X$, there is a unique k-linear map $\lambda : V \otimes W \to X$ such that $\langle v, w \rangle = \lambda(v \otimes w)$.

"*[A]ll the proofs I came up with involved the universal property of tensor products, never the elements themselves. It was incredibly unsatisfying, it was like I could only describe the outside of an alien world instead of getting to know its inhabitants.*" – ibid.

(137) Let V_1, V_2, W_1, W_2 be k-vector spaces and $\alpha : V_1 \to V_2$ and $\beta : W_1 \to W_2$ be k-linear maps. Show that there is a unique linear map $\alpha \otimes \beta : V_1 \otimes W_1 \to V_2 \otimes W_2$ such that $(\alpha \otimes \beta)(v \otimes w) = \alpha(v) \otimes \beta(w)$.

(138) Let V_1, V_2, V_3, W_1, W_2, W_3 be k-vector spaces and $\alpha_1 : V_1 \to V_2$, $\alpha_2 : V_2 \to V_3$, $\beta_1 : W_1 \to W_2$ and $\beta_2 : W_2 \to W_3$ be k-linear maps. Show that $(\alpha_2 \otimes \beta_2) \circ (\alpha_1 \otimes \beta_1) = (\alpha_2 \circ \alpha_1) \otimes (\beta_2 \circ \beta_1)$.

At this point, we have the basic formal properties to work with tensor products, but we have almost no ability to compute with them. For example, we don't even know a basis for $k^m \otimes k^n$! We turn to this issue next.

(139) Let I be a set of vectors spanning V and let J be a set of vectors spanning W. Show that the tensor products $v \otimes w$, for $v \in I$ and $w \in J$, span $V \otimes W$.

(140) Let U be a vector space and let I be a linearly independent subset of U. Prove that there is a basis B of U containing I. This will require Zorn's Lemma. [1]

(141) Let U be a vector space, let I be a linearly independent subset of U and let $u \in I$. Show that there is a linear form $\alpha : U \to k$ such that $\alpha(u) = 1$ and $\alpha(u') = 0$ for $u' \in I \setminus \{u\}$.

(142) Let I be a linearly independent subset of V and let J be a linearly independent subset of W. Show that the tensor products $v \otimes w$, for $v \in I$ and $w \in J$, are linearly independent in $V \otimes W$.

(143) Let I be a basis of V and let J be a basis of W. Show that the tensor products $v \otimes w$, for $v \in I$ and $w \in J$, are a basis of $V \otimes W$.

That was a lot of abstraction, so let's do something concrete.

(144) Let α and β be the linear maps $\mathbb{R}^2 \to \mathbb{R}^2$ given by the matrices $\left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 5 & 6 \\ 7 & 8 \end{smallmatrix}\right]$. Choose a basis for $\mathbb{R}^2 \otimes \mathbb{R}^2$ and write down the matrix of $\alpha \otimes \beta$.

"*After a few months, though, I realized something. I hadn't gotten any better at understanding tensor products, but I was getting used to not understanding them. It was pretty amazing. I no longer felt anguished when tensor products came up; I was instead almost amused by their cunning ways.*" – ibid.

[1] Although Problems 140 and 141 genuinely use the Axiom of Choice, Problems 142 and 143 are true without it. Here is a sketch of a proof. Note that the arguments suggested in this worksheet work fine in finite dimensional vector spaces. Now, let V and W be vector spaces of any dimension, let I and J be linearly independent subsets of V and W and suppose for the sake of contradiction that there is a linear relation $\sum c_{vw} v \otimes w$ between elements $v \otimes w$ as above. Note that this linear relation involves only *finitely* many elements of I and J. Moreover, the deduction of this dependence from the relations $(*)$ must also use only finitely many elements of V and W. Let $\overline{V}$ and $\overline{W}$ be the subspaces of V and W spanned by these finitely many elements. Then we obtain a counterexample to Problem 142 inside $\overline{V} \otimes \overline{W}$, and we have $\dim \overline{V}, \dim \overline{W} < \infty$.