UNIQUE FACTORIZATION DOMAINS (UFDS)

Vocabulary: irreducible element, prime element, Unique Factorization Domain, UFD.

Definition. A ring R is called a ***domain*** provided that R is nonzero and for all $a, b \in R$ we have $ab = 0$ implies $a = 0$ or $b = 0$. A commutative domain is called an ***integral domain***.

Definition. Let R be an integral domain and let r be an element of R. We say that r is ***composite*** if r is nonzero and r can be written as a product of two non-units. We say that r is ***irreducible*** if it is neither composite, nor 0, nor a unit.

Thus every element of R is described by precisely one of the adjectives "zero", "unit", "composite", "irreducible".

Definition. Let R be an integral domain and let r be an element of R. We say that r is ***prime*** if (r) is a prime ideal and $r \neq 0$.

(50) Show that prime elements are irreducible.

(51) Let k be a field and let $k[t^2, t^3]$ be the subring of $k[t]$ generated by t^2 and t^3.
- (a) Check that t^2 is irreducible in $k[t^2, t^3]$.
- (b) Show that t^2 is not prime in $k[t^2, t^3]$.

(52) Consider the subring $\mathbb{Z}[\sqrt{-13}]$ of $\mathbb{C}$.
- (a) Show that 7 is irreducible in $\mathbb{Z}[\sqrt{-13}]$. (Hint: Complex absolute value.)
- (b) Show that 7 is not prime in $\mathbb{Z}[\sqrt{-13}]$.

Definition. A ***Unique Factorization Domain*** or ***UFD*** is an integral domain R is which every nonzero, nonunity $r \in R$ has the following properties:
- (factorization) r can be written as a finite product of (not necessarily distinct) irreducibles p_i of R: $r = p_1 p_2 \cdots p_n$.
- (uniqueness of factorization) if $r = q_1 q_2 \cdots q_m$ is another factorization of r into irreducibles then $m = n$ and there exists $\sigma \in S_n$ so that $p_j \in q_{\sigma(j)} R^\times$ for $1 \leq j \leq n$.

In plain language, in a UFD every non-zero non-unit can be written uniquely (up to reordering and unit multiple) as a product of irreducible elements.

(53) Show that $\mathbb{Z}[\sqrt{-13}]$ and the ring $k[t^2, t^3]$ are not UFDs, by giving elements with two factorizations.

(54) Show that, in a UFD, irreducible elements are prime.

Definition. If R is a commutative domain and X is a subset of R, we define an element g of R to be a ***greatest common divisor*** or ***GCD*** of X
- if g divides every element of X and
- if h divides every element of X, then h divides g.

(55) Let X be a subset of R. Show that, if g and g' are both GCD's of X, then there is a unit u such that $g' = gu$.

(56) Show that, if R is a UFD, then every subset of R has a GCD.

(57) Let R be a ring where $\{x, y\}$ has a GCD for any x and $y \in R$. Show that irreducible elements of R are prime. Hint: Suppose that p is irreducible and $p|ab$. Consider $\mathrm{GCD}(pb, ab)$.

(58) Let R be a domain. Show that any element of R can be written in **at most** one way as a product of **prime** elements. (Uniqueness is up to reordering and multiplication by units, as in the definition of a UFD.)

Combining Problems 57 and 58, we see that, in a ring where any two elements have a GCD, every element has **at most** one **irreducible** factorization. We now have to address the question of when irreducible factorizations exist. It turns out that this holds in every Noetherian integral domain.

(59) Let R be a Noetherian integral domain. Show that there does not exist a sequence q_1, q_2, q_3, … of elements of R such that q_{j+1} divides q_j and q_j does not divide q_{j+1}.

(60) Let R be a Noetherian integral domain. Show that elements of R have **at least** one **irreducible** factorization. [1]

Putting together everything we have seen:

Theorem: Noetherian integral domain is a UFD, if and only if every two elements have a GCD, if and only if every subset has a GCD.

[1] The very careful student will notice a use of the Axiom of Choice here.