10. SOLVABLE GROUPS

Now that we have the Jordan-Holder theorem, we can start to classify groups according to what kind of factors appear in their composition/subnormal series. A basic example of this is the solvable groups:

Definition: A group G is called ***solvable*** if it has a subnormal series $1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_N = G$ such that G_j/G_{j-1} is abelian.

Problem 10.1. Show that S_3 and S_4 are solvable.

Problem 10.2. Show that a subgroup of a solvable group is solvable.

Problem 10.3. Show that a quotient group of a solvable group is solvable.

Problem 10.4. Show that, if $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ is a short exact sequence, and A and C are solvable, then B is solvable.

Problem 10.5. Show that a finite group is solvable if and only if all its Jordan-Holder factors are cyclic of prime order.

There is a standard algorithm to test whether a group is solvable, using the ***derived series***.

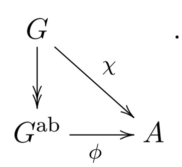
Definition: Let G be a group. The ***commutator subgroup***, also called the ***derived subgroup***, is the group generated by all products $ghg^{-1}h^{-1}$ for g and $h \in G$. It can be denoted $D(G)$ or $[G, G]$.

Problem 10.6. Show that $D(G)$ is normal in G.

Problem 10.7. Show that $G/D(G)$ is abelian.

Definition: The quotient $G/D(G)$ is called the ***abelianization*** of G and denoted G^{ab}.

Problem 10.8. Prove the ***universal property of the abelianization***: If G is a group, A is an abelian group and $\chi : G \to A$ is a group homomorphism, then there is a unique homomorphism $\phi : G^{\mathrm{ab}} \to A$ such that the diagram below commutes:



Definition: The ***derived series*** of G is the chain of subgroups $G \trianglerighteq D(G) \trianglerighteq D(D(G)) \trianglerighteq \cdots$. We'll denote the k-th group in this chain as $D_k(G)$.

Problem 10.9. Show that G is solvable if and only if there is some N for which $D_N(G) = \{e\}$.

Problem 10.10.
(1) For $n \geq 2$, show that $S_n^{\mathrm{ab}} \cong \{\pm 1\}$.
(2) For $n \geq 5$, show that A_n^{ab} is trivial.
(3) For $n \geq 5$, show that S_n is not solvable.