11. Direct products

Before starting our main topic, we want some lemmas about the following definition:

Definition: Let B be a group and let A and C be subgroups. Then AC is the set $\{ac : a \in A,\ c \in C\}$.

Problem 11.1. Show that, if B is a group and A and C are subgroups with $A \cap C = \{e\}$, then the map of sets $A \times C \to AC$ by $(a, c) \mapsto ac$ is a bijection.

Problem 11.2. Give an example of a group B and two subgroups A and C such that AC is not a subgroup of B. (Hint: There is an example in S_3.)

In light of the Jordan-Holder theorem, it is natural to ask, given two groups A and C, how we can put them together into a short exact sequence $1 \to A \to B \to C \to 1$. The most basic way to do this is by a direct product. As with modules and vector spaces, these come in both internal and external versions. I'll underline the internal products for this worksheet, but the usual notation is to use $\times$ for both of them.

Definition: Given two groups A and C, the ***direct product*** is the group whose underlying set is $A \times C$, with multiplication structure $(a_1, c_1) * (a_2, c_2) = (a_1 a_2, c_1 c_2)$.

Problem 11.3. Let B be a group which has two normal subgroups A and C, such that $A \cap C = \{e\}$ and such that $B = AC$.

(1) Show that, for any $a \in A$ and $c \in C$, we have $ac = ca$. (Hint: Think about $aca^{-1}c^{-1}$.)
(2) Show that B is isomorphic to the direct product $A \times C$.

We will write $B = A \underline{\times} C$ in this case when A and C are as above. This is an "internal direct product".

Problem 11.4. Show that $\mathrm{GL}_3(\mathbb{R}) = \mathrm{SL}_3(\mathbb{R}) \underline{\times} \mathbb{R}^\times \mathrm{Id}_3$.

Problem 11.5. Let $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ be a short exact sequence and suppose that there is a group homomorphism $\rho : B \to A$ with $\rho \circ \alpha = \mathrm{Id}$. In this case, we will say that the sequence $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ is ***left split***.

(1) Show that $B = \alpha(A) \underline{\times} \mathrm{Ker}(\rho)$.
(2) Show that $\alpha(A) \cong A$ and $\mathrm{Ker}(\rho) \cong C$, so $B \cong A \times C$.

Problem 11.6. For any groups A and C, show that there is a left split short exact sequence

$$1 \to A \to A \times C \to C \to 1.$$