12. SEMIDIRECT PRODUCTS

Once again, we ask how we can stick groups A and C together into a short exact sequence $1 \to A \to B \to C \to 1$. After direct products, the next most basic way is semidirect products. This time, we'll do the internal version first. Again, I'll underline the internal version for this worksheet, but the standard notation is to use the same symbol for both. We recall the definition:

> **Definition:** Let B be a group and let A and C be subgroups. Then AC is the set $\{ac : a \in A,\ c \in C\}$.

We proved last time that, if $A \cap C = \{e\}$, then the map $(a, c) \mapsto ac$ is a bijection from $A \times C$ to AC.

Problem 12.1. Let B be a group, let A be a **normal** subgroup of B and let C be any subgroup of B. Show that AC is a subgroup of B.

> **Definition:** Let B be a group, let A be a normal subgroup of B and let C be any subgroup of B. Suppose that $A \cap C = \{e\}$ and that $B = AC$. Then we say that B is the ***internal semidirect product*** of A and C and write $B = A\underline{\rtimes}C$.

Problem 12.2. Show that $S_3 = A_3\underline{\rtimes}S_2$, with S_2 embedded as the permutations that fix 3.

Problem 12.3. Let $B = A\underline{\rtimes}C$. Define a map $\phi : C \to \mathrm{Aut}(A)$ by $\phi(c)(a) = cac^{-1}$.

(1) Show that $\phi(c)$ is, as promised, an automorphism of A.
(2) Show that $\phi : C \to \mathrm{Aut}(A)$ is a group homomorphism.
(3) Show that
$$(a_1c_1)(a_2c_2) = \big(a_1\phi(c_1)(a_2)\big)(c_1c_2).$$

We use the formula in the last problem to define the external semidirect product:

> **Definition:** Let A and C be groups and let $\phi : C \to \mathrm{Aut}(A)$ be a group homomorphism. We define $A \rtimes_\phi C$ to be the group whose underlying set is $A \times C$, with multiplication
> $$(a_1, c_1)(a_2, c_2) = (a_1\phi(c_1)(a_2), c_1c_2).$$
> We sometimes omit ϕ when it is clear from context.

Problem 12.4. Check that $A \rtimes_\phi C$ is a group.

So Problem 12.3 says that, if $B = A\underline{\rtimes}C$, then $B \cong A \rtimes_\phi C$ for the action $\phi(c)(a) = cac^{-1}$.

Problem 12.5. Give two actions of C_2 on C_3 such that $S_3 \cong C_3 \rtimes C_2$ for one action and $C_6 \cong C_3 \rtimes C_2$ for the other.

Problem 12.6. Let p be prime. Show that $C_{p^2} \not\cong C_p \rtimes C_p$ for any action of C_p on C_p.

Problem 12.7. Let $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ be a short exact sequence and suppose that there is a group homomorphism $\sigma : C \to B$ with $\beta \circ \sigma = \mathrm{Id}$. In this case, we will say that the sequence $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ is ***right split***.

(1) Show that $B = \alpha(A)\underline{\rtimes}\sigma(C)$.
(2) Show that $\alpha(A) \cong A$ and $\sigma(C) \cong C$, so $B \cong A \rtimes C$.

Problem 12.8. For any groups A and C, and any action of C on A, show that there is a right split short exact sequence
$$1 \to A \to A \rtimes_\phi C \to C \to 1.$$