13. ABELIAN EXTENSIONS

Here is a lemma from the homework; check that everyone in your group solved it.

Problem 13.1. Let $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ be short exact. Let $\tilde{C}$ be any subset of G such that $\beta : \tilde{C} \to C$ is bijective. Then every $b \in B$ can be uniquely written in the form $\alpha(a)\tilde{c}$ for $a \in A$ and $\tilde{c} \in \tilde{C}$.

In this worksheet, we will study short exact sequences $1 \to A \to G \to H \to 1$ with A abelian; when such a short exact sequence exists, we say that G is an ***abelian extension*** of H. A special case is when A is central in G, in this case, we say that G is a ***central extension*** of H.

Problem 13.2. Let $1 \to A \to G \to H \to 1$ be an abelian extension. Since A is normal in G, we get an action of G on A by $g : a \mapsto gag^{-1}$. Show that the map $G \to \mathrm{Aut}(A)$ factors through H.

We'll write $\phi : H \to \mathrm{Aut}(A)$ for the resulting action.

Problem 13.3. Show that the action ϕ is trivial (meaning $\phi(h)(a) = a$ for all $h \in H$ and $a \in A$) if and only if the extension $1 \to A \to G \to H \to 1$ is central.

Classifying abelian extensions with fixed (A, H) thus comes down to two parts (1) classifying all actions of H on A and (2) for each action ϕ, classifying all abelian extensions that result. We know there is always at least one such extension: the semidirect product $A \rtimes_\phi H$.

Problem 13.4. Let p be a prime number, let H be a group of order p^k and let $1 \to C_p \to G \to H$ be an abelian extension. Show that it must be a central extension.

Problem 13.5. Let n be a positive integer and let $1 \to Z \to G \to C_n \to 1$ be a **central** extension. Show that G is abelian. (Hint: Let $g \in G$ map to a generator of C_n. Use Problem 13.1 with $S = \{1, g, g^2, \dots, g^{n-1}\}$.)

Problem 13.6. Let p be a prime number and let G be a group of order p^k. Show that G lies in a central extension $1 \to C_p \to G \to H \to 1$ for some H of order p^{k-1}.

Problem 13.7. Let p be prime. Show that every group of order p^2 is isomorphic to C_p^2 or C_{p^2}.

Problem 13.8. Let p and q be distinct prime numbers, let $A \cong C_p$, $H \cong C_q$ and let $1 \to A \to G \to H \to 1$ be an abelian extension.

(1) If $p \not\equiv 1 \bmod q$, show that the action of H on A is trivial.
(2) If the action of H on A is trivial, show that $G \cong C_{pq} \cong C_p \times C_q$.
(3) If the action ϕ of H on A is nontrivial, show that $G \cong C_p \rtimes_\phi C_q$.

Problem 13.9. Let p be an odd prime, let $A \cong C_p$, $H \cong C_p^2$. In this problem, we will classify abelian extensions $1 \to A \to G \to H \to 1$. We write z for a generator of A and $\tilde{x}$ and $\tilde{y}$ for lifts of x and y to G.

(1) Show that z is central in G. (Hint: What can ϕ be?)
(2) Show that every element of G is uniquely of the form $\tilde{x}^a\tilde{y}^b z^c$ for $a, b, c \in \{0, 1, \dots, p-1\}$.
(3) Show that $\tilde{x}^p$, $\tilde{y}^p$ and $\tilde{y}\tilde{x}\tilde{y}^{-1}\tilde{x}^{-1}$ are of the form z^i, z^j and z^k for some i, j and $k \in \mathbb{Z}/p\mathbb{Z}$.
(4) Suppose that $k = 0$. Show that G is abelian and is isomorphic to either C_p^3 or $C_{p^2} \times C_p$.
(5) Suppose that $k \neq 0$ and $(i, j) = (0, 0)$. Show that $(\tilde{x}^{a_1}\tilde{y}^{b_1}z^{c_1})(\tilde{x}^{a_2}\tilde{y}^{b_2}z^{c_2}) = \tilde{x}^{a_1+a_2}\tilde{y}^{b_1+b_2}z^{c_1+c_2+kb_1a_2}$. Show that G is isomorphic to the group of matrices of the form $\begin{bmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{bmatrix}$ with entries in $\mathbb{Z}/p\mathbb{Z}$.
(6) Suppose that $(i, j) \neq (0, 0)$. Show that there are a and b not both $0 \bmod p$ such that $(\tilde{x}^a\tilde{y}^b)^p = 1$. This is where you will need that p is odd.
(7) Suppose that $(i, j) \neq (0, 0)$ Show that $G \cong C_{p^2} \rtimes_\phi C_p$ and describe the action of C_p on C_{p^2}.

Problem 13.10. Let p be an odd prime. Show that every group of order p^3 is isomorphic to one of

$$C_p^3, \quad C_{p^2} \times C_p, \quad C_{p^3}, \quad C_{p^2} \rtimes C_p, \quad \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{Z}/p\mathbb{Z} \right\}.$$