14. THE SYLOW THEOREMS

Let p be a prime.

Definition: A ***p-group*** is a group P with $\#(P) = p^k$ for some k. For a group G, a ***p-subgroup*** of G is a subgroup which is a p-group.

Problem 14.1. Let P be a p group and let X be a finite set on which P acts. Suppose that $\#(X) \not\equiv 0 \bmod p$. Show that P fixes some point of X.

Let G be a group. Factor $\#(G)$ as $p^k m$ where p does not divide m.

Definition: A ***Sylow p-subgroup*** of G is a subgroup of G of order p^k.

Problem 14.2. Let Γ be a finite group with a Sylow p-subgroup Π. Let G be a subgroup of Γ.

(1) Show that G has a Sylow p-subgroup P. Hint: Consider G acting on Γ/Π.
(2) Show, more specifically, that there is some $\gamma \in \Gamma$ such that $P = G \cap \gamma \Pi \gamma^{-1}$.

Hint for the following three problems: Use Problem 14.2.

Problem 14.3. (**The first Sylow theorem**) Show that every finite group G has a Sylow p-subgroup.

Problem 14.4. Let G be a finite group and let P be a Sylow p-subgroup with $\#(P) = p^k$.

(1) Let Q be a p-subgroup of G. Show that there is some $g \in G$ such that $Q \subseteq gPg^{-1}$.
(2) Let H be a subgroup of G whose order is divisible by p^k. Show that there is some $g \in G$ such that $H \supseteq gPg^{-1}$.

Problem 14.5. (**The second Sylow theorem**) Let G be a finite group and let P_1 and P_2 be two Sylow p-subgroup of G. Show that there is some $g \in G$ such that $P_2 = gP_1g^{-1}$.

Let G be a group and let H be a subgroup of G. We define $N_G(H) = \{g \in G : gHg^{-1} = H\}$. The group $N_G(H)$ is called the ***normalizer*** of H in G.

Problem 14.6. Map $G/N_G(P)$ to the set of Sylow p-subgroups by sending the coset $gN_G(P)$ to gPg^{-1}. Show that this map is well defined, and is a bijection.

Problem 14.7.
(1) Show that P is normal in $N_G(P)$.
(2) Let Q be a p-subgroup of $N_G(P)$. Show that $Q \subseteq P$.
(3) Let H be a p-subgroup of G. Show that $H \cap N_G(P) = H \cap P$.

Problem 14.8. Since P is a subgroup of G, the group P acts on $G/N_G(P)$. Show that the only coset which is fixed for this action is $eN_G(P)$.

Problem 14.9. (**The third Sylow theorem**) The number of Sylow p-subgroups of G is $\equiv 1 \bmod p$.