15. Some problems with Sylow groups

Problem 15.1. Let G be a group of order $p^k m$ where p does not divide m. Show that the number of p-Sylow subgroups of G divides m.

Problem 15.2. Let G and H be finite groups and p a prime number. Let P and Q be p-Sylow subgroups of G and H.

(1) Show that $P \times Q$ is a p-Sylow subgroup of $G \times H$.
(2) Show that every p-Sylow subgroup of $G \times H$ is of the form $P' \times Q'$ for P' and Q' p-Sylow subgroups of G and H.

Problem 15.3. Let $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ be a short exact sequence of finite groups and let Q be a p-Sylow subgroup of B. Show that $\alpha^{-1}(Q)$ and $\beta(Q)$ are p-Sylow subgroups of A and C respectively.

Problem 15.4. Let $p < q$ be primes and let G be a group of order pq.

(1) Show that the q-Sylow subgroup of G is normal.
(2) Conclude that there is a short exact sequence $1 \to C_q \to G \to C_p \to 1$.
(3) Show that $G \cong C_q \rtimes C_p$ for some action of C_p on C_q.

Problem 15.5. Show that there are no simple groups of order 40. (Hint: Look at 5-Sylows.)

Problem 15.6. In this problem, we will show that there is no simple group G of order 80.

(1) Show that, if G were such a group, then G would have five 2-Sylow subgroups.
(2) Consider the map $G \to S_5$ to get a contradiction.

Problem 15.7. A standard rite of passage is to check that there are no non-abelian simple groups of order < 60, so let's do that. Let G be a non-abelian simple group.

(1) Show that the order of G is not a prime power.
(2) Show that, for every prime p dividing $\#(G)$, there must be some n_p dividing $\#(G)$ with $n_p > 1$ and $n_p \equiv 1 \bmod p$.

At this point, we have ruled out all cases except 12, 24, 30, 36, 48 and 56.

(3) In the notation of the previous problem, show that furthermore we must have $\#(G) | \frac{n_p!}{2}$.

This rules out 12, 24, 48 (take $p = 2$) and 36 (take $p = 3$).

(4) Suppose that G were a simple group of order 30. Show that G would contain 24 elements of order 5 and 20 elements of order 3; deduce a contradiction.
(5) Suppose that G were a simple group of order 56. Show that G would contain 48 elements of order 7 and > 8 elements whose order is a power of 2; deduce a contradiction.