16. Review of polynomial rings

Throughout this worksheet, let k be a field. Let $k[x]$ be the ring of polynomials with coefficients in k.

Here are some things that you hopefully know, and may use without proof.

- Let $b(x) \in k[x]$ be a nonzero polynomial of degree d. Let $a(x)$ be any polynomial in $k[x]$. Show that there are unique polynomials $q(x)$ and $r(x)$, with $\deg r < d$, such that
$$a(x) = b(x)q(x) + r(x).$$
- The ring $k[x]$ is Euclidean, is a PID and a UFD.
- If $p(x)$ is an irreducible polynomial, then $p(x)k[x]$ is a maximal ideal, and $k[x]/p(x)k[x]$ is a field.

Problem 16.1. Let $b(x) \in k[x]$ be a nonzero polynomial of degree d. Show that the ring $k[x]/b(x)k[x]$ is a k-vector space of dimension d.

Let K be a larger field containing k. For $\theta \in K$, we say that θ is ***algebraic*** over k if there is a nonzero polynomial $f(t)$ in $k[t]$ with $f(\theta) = 0$.

Problem 16.2. Let $\theta \in K$ be algebraic over k. Let $I \subset k[t]$ be $\{f(t) \in k[t] : f(\theta) = 0\}$.

(1) Show that $I = m(t)k[t]$ for some irreducible polynomial m.
(2) Show that $k[\theta]$, meaning the subring of K generated by k and θ, is isomorphic to $k[t]/m(t)k[t]$.

The polynomial $m(t)$ is called the ***minimal polynomial*** of θ.

Problem 16.3. Let K be a larger field containing k. Let α and β be two algebraic elements of K which have the same minimal polynomial. Show that there is an isomorphism $k[\alpha] \to k[\beta]$ taking α to β.

Problem 16.4. Show that θ is algebraic over k if and only if $\dim_k k[\theta] < \infty$.

Problem 16.5. Show that the set of elements of K which are algebraic over k is a subfield of K.