Definition: Let k be a field, let $f(x)$ be a polynomial in $k[x]$ and let K be an extension field of f. We will say that f ***splits in*** K if f factors as a product of linear polynomials in $K[x]$. We say that K is a ***splitting field of*** f if f splits as a product $c\prod(x-\theta_j)$ in $K[x]$ and the field K is generated by k and by the θ_j.

For example, if $k=\mathbb{Q}$ and $\theta_1, \theta_2, \dots, \theta_n$ are the roots of $f(x)$ in $\mathbb{C}$, then $\mathbb{Q}[\theta_1, \dots, \theta_n]$ is a splitting field of $f(x)$.

Problem 18.1. Let k be a field and let $f(x)$ be a polynomial in $k[x]$. Show that f has a splitting field.

Problem 18.2. Let L be a splitting field for x^3-2 over $\mathbb{Q}$. Show that $[L:\mathbb{Q}]=6$. (Hint: At one point, it will be very useful to use the fact that $\mathbb{Q}[\sqrt[3]{2}]$ is a subfield of $\mathbb{R}$.)

Problem 18.3. Let $L=\mathbb{C}(x_1, x_2, \dots, x_n)$. Let e_k be the k-th elementary symmetric polynomial and let $K=\mathbb{C}(e_1, e_2, \dots, e_n)\subset L$. Show that L is a splitting field for $x^n-e_1x^{n-1}+e_2x^{n-2}-\cdots\pm e_n$ over K.

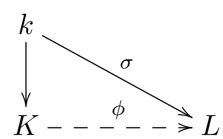
Problem 18.4. Let

$$f(x)=\left(x-\cos\tfrac{2\pi}{7}\right)\left(x-\cos\tfrac{4\pi}{7}\right)\left(x-\cos\tfrac{8\pi}{7}\right)=\tfrac{1}{8}\left(8x^3+4x^2-4x-1\right).$$

I promise, and you may trust me, that $f(x)$ is irreducible. Let $K=\mathbb{Q}(\cos\frac{2\pi}{7})$.

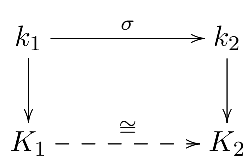
(1) Show that $[K:\mathbb{Q}]=3$.
(2) Show that $f(x)$ splits in K. Hint: Use the double angle formula.
(3) Show that there is an automorphism $\sigma:K\to K$ with $\sigma(\cos\frac{2\pi}{7})=\cos\frac{4\pi}{7}$.

Problem 18.5. Let k be a field and let $f(x)$ be a polynomial in $k[x]$. Let K be a splitting field of f in which f splits as $\prod(x-\alpha_j)$. Let $\sigma:k\to L$ be a field homomorphism and let $\sigma(f):=\sum\sigma(f_j)x^j$ split in L. Show that there is an injection $\phi:K\to L$ making the diagram



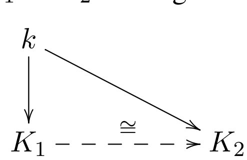
commute. Hint: Think about $k\subseteq k[\alpha_1]\subseteq k[\alpha_1,\alpha_2]\subseteq\cdots\subseteq k[\alpha_1,\alpha_2,\dots,\alpha_n]=K$.

Problem 18.6. Let k_1 and k_2 be two fields and let $\sigma:k_1\to k_2$ be an isomorphism. Let $(x)=\sum f_jx^j$ be a polynomial in $k_1[x]$ and let $\sigma(f)(x):=\sum\sigma(f_j)x^j$. Let K_1 be a splitting field of f and let K_2 be a splitting field of $\sigma(f)$. Show that there is an isomorphism $K_1\cong K_2$ making the diagram



commute.

Problem 18.7. Let k be a field and let $f(x)$ be a polynomial in $k[x]$. Let K_1 and K_2 be two splitting fields of f. Show that there is an isomorphism $K_1\cong K_2$ making the diagram



commute. **So splitting fields are unique.**