Definition: Let $K \subseteq L$ be fields. An ***automorphism*** of L is a bijection $\sigma : L \to L$ with $\sigma(x+y) = \sigma(x) + \sigma(y)$ and $\sigma(xy) = \sigma(x)\sigma(y)$. An ***automorphism of*** L ***fixing*** K is an automorphism of L obeying $\sigma(a) = a$ for all $a \in K$. We write $\operatorname{Aut}(L)$ for the automorphisms of L and $\operatorname{Aut}(L/K)$ for the automorphisms of L fixing K.

Problem 19.1. Let $K \subseteq L$ be fields. Let $f(x)$ be a polynomial in $K[x]$; let $\{\theta_1, \theta_2, \ldots, \theta_r\}$ be the roots of f in L.

(1) Show that $\operatorname{Aut}(L/K)$ maps $\{\theta_1, \theta_2, \ldots, \theta_r\}$ to itself.
(2) Show that stabilizer of θ_j in $\operatorname{Aut}(L/K)$ is $\operatorname{Aut}(L/K(\theta_j))$.
(3) Let $L = \mathbb{Q}(\sqrt[4]{2})$ and let $f(x) = x^2 - 2$. Show that the roots of $f(x)$ in L are $\{\pm\sqrt{2}\}$ and show that $\operatorname{Aut}(L/\mathbb{Q})$ fixes both of them.

Problem 19.2. . Let K be a field, let f be a polynomial in $K[x]$, let L be a splitting field for f and let $\{\theta_1, \theta_2, \ldots, \theta_n\}$ be the roots of f in L. Assume $\{\theta_1, \theta_2, \ldots, \theta_n\}$ are distinct.[1]

(1) Show that the action of $\operatorname{Aut}(L/K)$ takes $\{\theta_1, \theta_2, \ldots, \theta_n\}$ to itself.
(2) Show that this action of $\operatorname{Aut}(L/K)$ gives an **injection** $\operatorname{Aut}(L/K) \hookrightarrow S_n$.

Problem 19.3. Let K, f, L and $\{\theta_1, \theta_2, \ldots, \theta_n\}$ be as in Problem 19.2. Let $g(x)$ be an irreducible factor of $f(x)$ in $K[x]$ and renumber the θ's so that $\{\theta_1, \theta_2, \ldots, \theta_m\}$ are the roots of g in L. Show that $\{\theta_1, \theta_2, \ldots, \theta_m\}$ is the $\operatorname{Aut}(L/K)$-orbit of θ_1 in L. Hint: Apply Problem 18.6 to the diagram

$$\begin{array}{ccccc} K[\theta_i] & \xleftarrow{\cong} & K[x]/g(x)K[x] & \xrightarrow{\cong} & K[\theta_j] \\ \downarrow & & & & \downarrow \\ L & - - - - - - & - - - - - - & - - - \rightarrow & L \end{array}$$

Problem 19.4. Let L be the splitting field of $x^3 - 2$ over $\mathbb{Q}$. Show that $\operatorname{Aut}(L/\mathbb{Q}) \cong S_3$.

Problem 19.5. Let $L = \mathbb{Q}(\cos \frac{2\pi}{7})$. Show that $\operatorname{Aut}(L/\mathbb{Q}) \cong C_3$.

[1] This happens if and only if $\operatorname{GCD}(f(x), f'(x)) = 1$, see the homework.