1. SYMMETRIES OF POLYNOMIALS

Let S_n be the group of permutations of $1, 2, \dots, n$. For two permutations σ and τ, we will write either $\sigma \circ \tau$ or $\sigma\tau$ for the composition: $(\sigma\tau)(j) := \sigma(\tau(j))$. We will often write permutations using cycle notation: $(i_1 i_2 \cdots i_k)$ means the permutation which cycles $i_1 \mapsto i_2 \mapsto \cdots \mapsto i_k \mapsto i_1$ and fixes everything not in $\{i_1, i_2, \dots, i_k\}$. We will let S_n act on the ring of polynomials $\mathbb{C}[r_1, \dots, r_n]$ in the obvious way.

Set

$$\Delta = \prod_{i<j} (r_i - r_j).$$

Problem 1.1. (1) For any permutation σ in S_n, show that $\sigma(\Delta^2) = \Delta^2$.
(2) For any permutation σ in S_n, show that $\sigma(\Delta) = \pm\Delta$.

Let $\omega = \frac{-1+\sqrt{-3}}{2}$. When we studied the cubic formula, we set

$$P = r_1 + \omega r_2 + \omega^2 r_3.$$

Let A_3 be the subgroup $\{e,\ (123),\ (123)^2\}$ of S_3.

Problem 1.2. (1) For any permutation $\sigma \in A_3$, show that $\sigma(P^3) = P^3$.
(2) For any permutation $\sigma \in A_3$, show that $\sigma(P) = \omega^k P$ for some integer k.

We set $\epsilon(\sigma) = \frac{\sigma(\Delta)}{\Delta}$, so $\epsilon(\sigma) \in \{\pm 1\}$. For $\sigma \in A_3$, we set $\eta(\sigma) = \frac{\sigma(P)}{P}$, so $\eta(\sigma) \in \{1, \omega, \omega^2\}$.

Problem 1.3. Show that $\epsilon(\sigma\tau) = \epsilon(\sigma)\epsilon(\tau)$, for σ and $\tau \in S_n$. Show that $\eta(\sigma\tau) = \eta(\sigma)\eta(\tau)$, for σ and $\tau \in A_3$.

We generalize these two examples. Let f be a nonzero polynomial in $\mathbb{C}[r_1, r_2, \dots, r_n]$ and let m be a positive integer. Define

$$G = \{\sigma \in S_n : \sigma(f^m) = f^m\} \qquad H = \{\sigma \in S_n : \sigma(f) = f\}.$$

Problem 1.4. (1) For $\sigma \in G$, define $\chi_f(\sigma) = \frac{\sigma(f)}{f}$. Show that $\chi_f(\sigma)$ is an m-th root of unity in $\mathbb{C}^*$.
(2) Show that G and H are subgroups of S_n.
(3) Show that, for σ and $\tau \in G$, we have $\chi_f(\sigma\tau) = \chi_f(\sigma)\chi_f(\tau)$.

Here are polynomials related to the quartic formula:

f	f^m
$T = r_1 + r_2 - r_3 - r_4$	T^2
$U = r_1 - r_2 + r_3 - r_4$	U^2
$V = r_1 - r_2 - r_3 + r_4$	V^2
$T + \omega U + \omega^2 V$	$\left(T + \omega U + \omega^2 V\right)^3$

Problem 1.5. For each of the polynomials in the table above, describe G, H and χ_f.