21. Towers of field extensions and Galois groups

We recall from last time:

Theorem/Definition: Let L/F be a field extension of finite degree. The following are equivalent:

(1) For every $\theta \in L$, the minimal polynomial of θ over F is separable and splits in L.
(2) L is the splitting field of a separable polynomial $f(x) \in F[x]$.
(3) We have $\#\operatorname{Aut}(L/F) = [L : F]$.
(4) The fixed field of $\operatorname{Aut}(L/F)$ is F.

A field extension L/F which satisfies these equivalent definitions is called ***Galois***.

Theorem: Let L/F be a Galois extension. Let $\theta \in L$. Then the minimal polynomial of θ over F is $\prod_{\phi \in \operatorname{Gal}(L/F)\theta}(x - \phi)$.

Throughout this worksheet, let $F \subseteq K \subseteq L$ be field extensions of finite degree, with L/F Galois.

Problem 21.1. Show that L/K is Galois.

Problem 21.2. Show that $\operatorname{Gal}(L/K)$ is a subgroup of $\operatorname{Gal}(L/F)$.

Problem 21.3. Show that $\#\operatorname{Gal}(L/K) = [L : K]$ and $[\operatorname{Gal}(L/F) : \operatorname{Gal}(L/K)] = [K : F]$.

Problem 21.4. Show that the fixed field of $\operatorname{Gal}(L/K)$ is K.

Problem 21.5. Let L/F be a Galois extension. Show that the map

$$\{\text{fields } K \text{ with } F \subseteq K \subseteq L\} \longrightarrow \{\text{subgroups of } \operatorname{Gal}(L/F)\}$$

given by $K \mapsto \operatorname{Gal}(L/K)$ is injective. (In fact, it is bijective, but I don't think we have the toolkit to prove that yet.)

Problem 21.6. Let $\sigma \in \operatorname{Gal}(L/F)$. Show that $\operatorname{Gal}(L/\sigma(K)) = \sigma \operatorname{Gal}(L/K)\sigma^{-1}$ (as subgroups of $\operatorname{Gal}(L/F)$).

Problem 21.7. Show that the following are equivalent:

(1) The subgroup $\operatorname{Gal}(L/K)$ is normal in $\operatorname{Gal}(L/F)$.
(2) For all $\sigma \in \operatorname{Gal}(L/F)$, we have $\sigma(K) = K$.
(3) For all $\theta \in K$, the minimal polynomial of θ over F splits in K.
(4) K is the splitting field of a separable polynomial with coefficients in F.
(5) K/F is Galois.

Problem 21.8. In the situation above, show that we have a short exact sequence $1 \to \operatorname{Gal}(L/K) \to \operatorname{Gal}(L/F) \to \operatorname{Gal}(K/F) \to 1$.