22. Artin's lemma

The following problem was on the problem sets, check that everyone knows how to solve it:

Problem 22.1. Let L be a field, let H be a group of automorphisms of L and let $F = \mathrm{Fix}(H)$, the elements of L fixed by H. Suppose that V is an L-vector subspace of L^n and that H takes V to itself. Show that V contains a nonzero element of F^n.

One of several results called Artin's Lemma: Let L be a field, let H be a finite group of automorphisms of L and let $F = \mathrm{Fix}(H)$, the elements of L fixed by H. Then $[L : F] = \#(H)$ and $H = \mathrm{Aut}(L/F)$.

Throughout this worksheet, let L, H and F be as above.

Problem 22.2. Show that $\#(H) \leq [L : F]$. This is just quoting something you've already done.

Suppose for the sake of contradiction that there are $n > \#(H)$ elements $\alpha_1, \alpha_2, \ldots, \alpha_n \in L$ which are linearly independent over F. Define

$$V = \left\{ (c_1, c_2, \ldots, c_n) \in L^n \ : \ \sum_j c_j h(\alpha_j) = 0 \ \forall h \in H \right\}.$$

Problem 22.3. Show that V is an L-vector subspace of L^n and that H takes V to itself.

Problem 22.4. Show that $\dim_L V > 0$.

Problem 22.5. Deduce a contradiction, and explain why you have proved $[L : F] = \#(H)$.

Problem 22.6. Show that $H = \mathrm{Aut}(L/F)$.

Artin's Lemma gives us a wide source of Galois extensions:

Problem 22.7. Let L, H and F be as in Artin's Lemma. Show that $[L : F]$ is Galois.