2. Characters of the symmetric and alternating groups

We recall the map $\epsilon : S_n \to \{\pm 1\}$, defined by $\epsilon(\sigma) = \frac{\sigma(\prod(r_i - r_j))}{\prod(r_i - r_j)}$. You showed that $\epsilon(\sigma\tau) = \epsilon(\sigma)\epsilon(\tau)$. This means that ϵ is an example of a ***character***:

Let G be a group. A ***character*** of G is a map $\chi : G \to \mathbb{C}^*$ obeying $\chi(gh) = \chi(g)\chi(h)$. The ***kernel*** of χ is $\{g \in G : \chi(g) = 1\}$. We define the ***alternating group***, A_n, to be the kernel of ϵ.

Problem 2.1. Which of the following permutations are in A_4:

$$\mathrm{Id},\ (12),\ (123),\ (12)(34),\ (1234)?$$

Two elements, g and h of G, are called ***conjugate***, if there is an element $c \in G$ with $h = cgc^{-1}$.

Problem 2.2. Let g and h be conjugate and let $\chi : G \to \mathbb{C}^*$ be a character. Show that $\chi(g) = \chi(h)$.

A ***transposition*** is a permutation of the type (ij). A 3***-cycle*** is a permutation of the type (ijk).

Problem 2.3. Let (ij) and $(k\ell)$ be two transpositions in S_n. Show that (ij) and $(k\ell)$ are conjugate.

Problem 2.4. Let $n \geq 5$ and let (ijk) be a 3-cycle in A_n. Show that (ijk) and $(ijk)^{-1}$ are conjugate in A_n, meaning that there is a permutation $c \in A_n$ with $c(ijk)c^{-1} = (ijk)^{-1}$.

Problem 2.5.
(1) Show that any permutation in S_n is a product of transpositions.
(2) Show that any permutation in A_n is a product of 3-cycles.

Problem 2.6. Let $\chi : S_n \to \mathbb{C}^*$ be a character.

(1) Show that either $\chi((ij)) = 1$ for all transpositions $(ij) \in S_n$ or else $\chi((ij)) = -1$ for all transpositions $(ij) \in S_n$.
(2) Show that either $\chi(g) = 1$ for all permutations $g \in S_n$ or else $\chi(g) = \epsilon(g)$ for all permutations $g \in S_n$.

Problem 2.7. Let $\chi : A_n \to \mathbb{C}^*$ be a character.

(1) Show that, for every 3-cycle (ijk), we have $\chi((ijk)) \in \{1, \omega, \omega^2\}$.
(2) Assuming that $n \geq 5$, show that, for all 3-cycles (ijk), we have $\chi((ijk)) = 1$.
(3) Assuming that $n \geq 5$, show that, for all $g \in A_n$, we have $\chi(g) = 1$.