3. A weak version of unsolvability of the quintic

One of the highlights of this course will be the proof of the unsolvability of the quintic. This worksheet proves a weaker version of this result.

Let L be the field of rational functions $\mathbb{C}(r_1, r_2, \ldots, r_n)$. Define e_1, e_2, $\ldots$, e_n as the coefficients of the polynomial:

$$(x - r_1)(x - r_2)\cdots(x - r_n) = x^n - e_1 x^{n-1} + e_2 x^{n-2} - e_3 x^{n-3} + \cdots \pm e_n.$$

> **Theorem (Ruffini):** Starting from e_1, e_2, $\ldots$, e_n, it is impossible to obtain the elements r_1, r_2, $\ldots$, r_n of L by the operations $+$, $-$, $\times$, $\div$, $\sqrt[n]{\ }$, **under the condition that**, every time we take an n-th root, we must stay in L.

At any point in the computation, there will be some list of elements of L which we have computed so far. Call them θ_1, θ_2, θ_3, $\ldots$ where each θ_k is either

(1) An element of $\mathbb{C}(e_1, \ldots, e_n)$.
(2) Of the form $\theta_i + \theta_j$, $\theta_i - \theta_j$, $\theta_i \times \theta_j$ or θ_i / θ_j, for i, $j < k$.
(3) Of the form $\sqrt[n]{\theta_j}$ for $j < k$.

Let G_j be the subgroup of S_n fixing θ_1, θ_2, $\ldots$, θ_j.

Problem 3.1. (1) If $\theta_k \in \mathbb{C}(e_1, \ldots, e_n)$, show that $G_k = G_{k-1}$.
(2) If θ_k is of the form $\theta_i + \theta_j$, $\theta_i - \theta_j$, $\theta_i \times \theta_j$ or θ_i / θ_j, for i, $j < k$, show that $G_k = G_{k-1}$.
(3) If θ_k is of the form $\sqrt[n]{\theta_j}$ for $j < k$, show that there is a character $\chi : G_{k-1} \to \mathbb{C}^*$ with kernel G_k.

Deleting the duplicate groups, we obtain a chain of subgroups

$$S_n = G_0 \supsetneq G_1 \supsetneq G_2 \supsetneq \cdots$$

such that, for each $k \geq 1$, there is a character $\chi : G_{k-1} \to \mathbb{C}^*$ with kernel G_k.

Problem 3.2. Let $n \geq 2$. Show that the first step of the chain must be $S_n \supsetneq A_n$.

Problem 3.3. Let $n \geq 5$. Show that the chain ends at A_n.

Problem 3.4. Prove Ruffini's Theorem!