

5. GROUP ACTIONS

Definition: Let G be a group and let X be a set. An **action** of G on X is a map $*$: $G \times X \rightarrow X$ obeying $(g_1 * g_2) * x = g_1 * (g_2 * x)$ and $e * x = x$.

Depending on context, we may denote $*$ by $*$, \times , \cdot or no symbol at all. This notion of an action can also be called a “left action”; a “right action” is a map $*$: $X \times G \rightarrow X$ obeying $x * (g_2 * g_1) = (x * g_2) * g_1$.

Problem 5.1. Let $G \times X \rightarrow X$ be a left action of G on X . Define a map $X \times G \rightarrow X$ by $(x, g) \mapsto g^{-1}x$. Show that this is a right action of G on X .

Problem 5.2. Let S_X be the group of bijections $X \rightarrow X$, with the group operation of composition. Show that an action of G on X is the same as a group homomorphism $G \rightarrow S_X$.

Definition: Let G be a group which acts on a set X . For $x \in X$, the **stabilizer** $\text{Stab}(x)$ of x is $\{g \in G : g * x = x\}$. For $g \in G$, the **fixed points** $\text{Fix}(g)$ of g are $\{x \in X : g * x = x\}$.

Problem 5.3. With G , X and x as above, show that $\text{Stab}(x)$ is a subgroup of X .

Problem 5.4. Let G , X and x be as above and let $g \in G$. Show that $\text{Stab}(gx) = g \text{Stab}(x)g^{-1}$.

Definition: For G , X and x as above, the **orbit** of x , written Gx , is $\{gx : g \in G\}$.

Problem 5.5. (The Orbit-Stabilizer theorem) If G is finite, show that $\#(G) = \#(Gx)\#(\text{Stab}(x))$.

The set of orbits of G on X is denoted $G \backslash X$. If we have a right action, we write X/G .

Problem 5.6. (Burnside’s Lemma¹) Let G be a finite group and let X be a finite set on G acts. Show that

$$\frac{1}{\#G} \sum_{g \in G} \#\text{Fix}(g) = \#(G \backslash X).$$

Definition: Let G be a group and let H be a subgroup. Let H act on G by $h * g = hg$. The orbits of this action are called the **right cosets** of H in G . The **left cosets** are the orbits for the right action $G * H \rightarrow G$. The number of cosets of H in G is called the **index** of H in G and written $[G : H]$.

Problem 5.7. Show that G has a left action on the set G/H of left cosets, such that $g_1 * (g_2 H) = (g_1 * g_2)H$. Show that the stabilizer of the coset eH is H .

Problem 5.8. (Lagrange’s Theorem²) Let G be a finite group and let H be a subgroup. Show that $\#(H)$ divides $\#(G)$.

Problem 5.9. Let G be a finite group with $\#(G) = N$. Let $g \in G$ and let the group generated by g have n elements.

- (1) Show that n divides N .
- (2) Show that $g^N = 1$.

¹Proved by Ferdinand Georg Frobenius.

²Proved by Camille Jordan.