# 7. SIMPLE GROUPS

**Definition:** A group $G$ is called ***simple*** if $G$ has precisely two normal subgroups, $G$ and $\{1\}$.

We remark that the trivial group is not simple, since it only has one normal subgroup.

**Problem 7.1.** Prove or disprove: Let $G$ be simple and let $H$ be any group. For every group homomorphism $\phi : G \to H$, either $\phi$ is injective or else $\phi$ is trivial.

**Problem 7.2.** Prove or disprove: Let $G$ be any group and let $H$ be simple. For any group homomorphism $\phi : G \to H$, either $\phi$ is surjective or else $\phi$ is trivial.

**Problem 7.3.** Let $p$ be a prime. Show that $C_p$ (the cyclic group of order $p$) is simple.

**Problem 7.4.** In this problem we will show that $A_n$ is simple, for $n \geq 5$. Let $N$ be a nontrivial normal subgroup of $A_n$. Let $g$ be a non-trivial element of $N$.

(1) Show that there is some 3-cycle $(ijk)$ in $A_n$ which does not commute with $g$.

We set $h = g(ijk)g^{-1}(ijk)^{-1}$.

(2) Show that $h \in N$.
(3) Show that $h$ has one of the following cycle structures: $(abc)(def)$, $(abcde)$, $(ab)(cd)$, $(abc)$.
(4) Show that $N$ contains a 3-cycle. In the case where $h$ has cycle type $(ab)(cd)$, you'll need to use that $n \geq 5$. **This part is a nuisance, and you may want to skip ahead and come back to it.**
(5) Show that $N = A_n$.

After $C_p$ and $A_n$, the most important simple groups are the projective special linear groups. Let $F$ be a field. The group $\mathrm{SL}_n(F)$ is the group of $n \times n$ matrices with entries in $F$ and determinant $1$. Let $Z \subset \mathrm{SL}_n(F)$ be $\{\zeta \,\mathrm{Id}_n : \zeta \in F \text{ with } \zeta^n = 1\}$. The ***projective special linear group*** $\mathrm{PSL}_n(F)$ is defined to be $\mathrm{SL}_n(F)/Z$. The group $\mathrm{PSL}_n(F)$ is simple, except in the cases of $\mathrm{PSL}_2(\mathbb{F}_2)$ (which is isomorphic to $S_3$) and $\mathrm{PSL}_2(\mathbb{F}_3)$ (which is isomorphic to $A_4$). The proof that $\mathrm{PSL}_n(F)$ has a lot of good ideas in it, but it is too long to make a worksheet problem; it might appear as a bonus lecture.