Definition: A ***subnormal series*** of a group G is a chain of subgroups $G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft G_3 \triangleleft \cdots \triangleleft G_N \subseteq G$ where G_{j-1} is normal in G_j. A ***composition series*** is a subnormal series where $G_0 = \{e\}$, $G_N = G$ and each subquotient G_j/G_{j-1} is simple. A ***quasi-composition series*** is a composition series where $G_0 = \{e\}$, $G_N = G$ and each subquotient is either simple or trivial.

Problem 8.1. Show that a group which has a quasi-composition series has a composition series.

Problem 8.2. Show that every finite group has a composition series.

Problem 8.3. Show that S_4 has a composition series with subquotients C_2, C_2, C_3 and C_2.

Problem 8.4. Show that $\mathrm{GL}_2(\mathbb{F}_7)$ has a composition series with subquotients C_2, $\mathrm{PSL}_2(\mathbb{F}_7)$, C_2 and C_3. You may assume that $\mathrm{PSL}_2(\mathbb{F}_7)$ is simple. (For a field of characteristic $\neq 2$, the group $\mathrm{PSL}_2(F)$ is $\mathrm{SL}_2(F)/\pm\mathrm{Id}$. See the worksheet on simple groups for the definition of $\mathrm{PSL}_n(F)$ in general.)

Problem 8.5. Let $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ be a short exact sequence, and let $\{1\} = A_0 \subset A_1 \subset \cdots \subset A_a = A$ and $\{1\} = C_0 \subset C_1 \subset \cdots \subset C_c = C$ be composition series of A and C. Show that

$$\{1\} = \alpha(A_0) \subset \alpha(A_1) \subset \cdots \subset \alpha(A_a) = \beta^{-1}(C_0) \subset \beta^{-1}(C_1) \subset \cdots \subset \beta^{-1}(C_c) = B$$

is a composition series for B.

Problem 8.6. Let $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ be a short exact sequence and let $\{1\} = B_0 \subset B_1 \subset \cdots \subset B_b = B$ be a composition series of B.

(1) Show that $\{1\} = \alpha^{-1}(B_0) \subseteq \alpha^{-1}(B_1) \subseteq \cdots \subseteq \alpha^{-1}(B_b) = A$ is a quasi-composition series for A.
(2) Show that $\{1\} = \beta(B_0) \subseteq \beta(B_1) \subseteq \cdots \subseteq \beta(B_b) = C$ is a quasi-composition series for C.

We are setting up to prove the Jordan-Holder theorem for groups. Here is a useful lemma.

Problem 8.7. Let $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ be a short exact sequence and let B' be a normal subgroup of B. Set $A' = \alpha^{-1}(B)$ and $C' = \beta(B)$. You might find it useful to think of A as a subgroup of B, and A' as $A \cap B'$.

(1) Show that $1 \to A' \to B' \to C' \to 1$ is a short exact sequence.
(2) Show that $1 \to A/A' \to B/B' \to C/C' \to 1$ is a short exact sequence.