9. THE JORDAN-HOLDER THEOREM

We recall the definitions from last time:

Definition: A ***subnormal series*** of a group G is a chain of subgroups $G_0 \lhd G_1 \lhd G_2 \lhd G_3 \lhd \cdots \lhd G_N \subseteq G$ where G_{j-1} is normal in G_j. A ***composition series*** is a subnormal series where $G_0 = \{e\}$, $G_N = G$ and each subquotient G_j/G_{j-1} is simple. A ***quasi-composition series*** is a composition series where $G_0 = \{e\}$, $G_N = G$ and each subquotient is either simple or trivial.

Let G be a group with a composition series $\{e\} = G_0 \rhd G_1 \rhd \cdots \rhd G_N = G$. We define N to be the length of the composition series and write $N = \ell(G)$. For a simple group Γ and a composition series $G_\bullet$, we define $m(G_\bullet, \Gamma)$ to be the number of quotients G_j/G_{j-1} which are isomorphic to Γ. Our aim today is to prove

Theorem (Jordan-Holder): Let G be a group and let $G_\bullet$ and $G'_\bullet$ be two composition series for G. Then $\ell(G_\bullet) = \ell(G'_\bullet)$ and, for any simple group Γ, we have $m(G_\bullet, \Gamma) = m(G'_\bullet, \Gamma)$.

Let $1 \to A \xrightarrow{\alpha} B \to\xrightarrow{\beta} C \to 1$ be a short exact sequence of groups. Let $B_\bullet$ be a composition series for B. Recall that we proved on the previous worksheet that $\{1\} = \alpha^{-1}(B_0) \subseteq \alpha^{-1}(B_1) \subseteq \cdots \subseteq \alpha^{-1}(B_b) = A$ is a quasi-composition series for A and $\{1\} = \beta(B_0) \subseteq \beta(B_1) \subseteq \cdots \subseteq \beta(B_b) = C$ is a quasi-composition series for C.

Problem 9.1. With the above notations, let $A_\bullet$ and $C_\bullet$ be the composition series obtained from deleting duplicate entries from the quasi-composition series above.

(1) Show that $\ell(B_\bullet) = \ell(A_\bullet) + \ell(C_\bullet)$.
(2) For any simple group Γ, show that $m(B_\bullet, \Gamma) = m(A_\bullet, \Gamma) + m(C_\bullet, \Gamma)$.

At this point, you have enough to prove the Jordan-Holder theorem for finite groups, by induction on $\#(G)$.

Problem 9.2. Check the base case: Jordan-Holder holds for the trivial group.

Problem 9.3. Check also that Jordan-Holder holds for simple groups.

Problem 9.4. Suppose that G is a finite group which is neither simple nor trivial, and suppose that Jordan-Holder holds for all groups of size less than $\#(G)$. Show that Jordan-Holder holds for G. This completes the induction, for $\#(G) < \infty$.

The Jordan-Holder theorem is also true for infinite groups that have composition series! Proving this requires no big new ideas, but a little more finesse. Define $L(G) = \min \ell(G_\bullet)$, where the minimum is over all composition series for G. Note $L(G) = 0$ if and only if G is trivial, and $L(G) > 0$ for any nontrivial G.

Problem 9.5. Check that $L(G) = 1$ if and only if G is simple.

Problem 9.6. Let $1 \to A \to B \to C \to 1$ be a short exact sequence of groups.

(1) Show that $L(B) \geq L(A) + L(C)$.[1]
(2) If A and C are nontrivial, show that $L(B) > L(A)$ and $L(B) > L(C)$.

Problem 9.7. Prove the Jordan-Holder theorem by induction on $L(G)$.

[1] In fact, equality holds and you have the tools to show it, but you don't need this.