A. CENTER, CENTRAL SERIES AND NILPOTENT GROUPS

A particularly nice sort of subnormal series is a central series, and a particularly nice kind of solvable group is a nilpotent group.

Definition: The ***center*** of a group G is the set $Z(G) := \{h : \ gh = hg \ \forall g \in G\}$.

Problem A.1. Let G be a group.

(1) Check that $Z(G)$ is a normal subgroup of G.
(2) Check that every subgroup of $Z(G)$ is normal in G.

Problem A.2. Let k be a field and let U be the group of matrices with entries in k of the form $\left[\begin{smallmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{smallmatrix}\right]$. Show that the center of U is the group of matrices of the form $\left[\begin{smallmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$.

This problem was on the problem sets; check that everyone in your group remembers how to do it.

Problem A.3. Let p be a prime and let G be a group of order p^k for $k \geq 1$. Show that $Z(G)$ is nontrivial.

Definition: Let G be a group. A ***central series*** of G is a sequence of subgroups $G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_N$ such that, if $g \in G$ and $h \in G_i$ then $ghg^{-1}h^{-1} \in G_{i-1}$, for $1 \leq i \leq N$. G is called ***nilpotent*** if it has a central series $G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_N$ with $G_0 = \{e\}$ and $G_N = G$.

Problem A.4. Let $G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_N$ be a series of subgroups of G. Show that G is a central series if and only if all the G_i are normal in G, and $G_i/G_{i-1} \subseteq Z(G/G_{i-1})$ for $1 \leq i \leq N$.

Problem A.5. Let k be a field and let U be the group of matrices with entries in k of the form

$$\begin{bmatrix} 1 & * & * & \cdots & * \\ & 1 & * & \cdots & * \\ & & 1 & \cdots & * \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}.$$

Show that U is nilpotent.

Problem A.6. Let p be a prime and let G be a group of order p^k for some $k \geq 1$. Show that G is nilpotent.

There is a converse to Problem A.6 which I hope to prove: The finite nilpotent groups are precisely the direct products of p-groups.

Problem A.7. Show that a nilpotent group is solvable.

Problem A.8. Show that a subgroup of a nilpotent group is nilpotent.

Problem A.9. Show that a quotient of a nilpotent group is nilpotent.