B. Finite nilpotent groups are products of p-groups.

Today's goal is to show:

Theorem: Let G be a finite group. Then G is nilpotent if and only if it is a direct product of p-groups.

Problem B.1. Show the easy direction: A direct product of p-groups is nilpotent.

From now on, let G be a finite nilpotent group with $\#(G) = \prod p_i^{k_i}$. We will be proving, by induction on $\#(G)$, that G is the direct product of its Sylow subgroups.

Problem B.2. Show that G has a central subgroup Z which is cyclic of prime order.

Let $G' = G/Z$, so we have a short exact sequence $1 \to Z \to G \xrightarrow{\beta} G' \to 1$. Let P_i' be a p_i-Sylow of G'. **By induction,** $G' = \prod_i P_i'$**.** We number the prime factors of $\#(G)$ such that $\#(Z) = p_1$. We analyze the Sylows of G, starting with the p_1-Sylow, and then the others.

Problem B.3.
(1) Show that $\beta^{-1}(P_1')$ is normal in G.
(2) Show that $\beta^{-1}(P_1')$ is a p_n-Sylow of G.

Problem B.4. Now, let $i > 1$. We have a short exact sequence $1 \to Z \to \beta^{-1}(P_i') \to P_i' \to 1$.

(1) Show that $\beta^{-1}(P_i')$ is normal in G.
(2) Show that the p_i-Sylow of $\beta^{-1}(P_i')$ is also a p_i-Sylow of G.
(3) Show that $\beta^{-1}(P_i') \cong Z \times P_i'$ (here is where you use Schur-Zassenhaus).
(4) Show that the p_i-Sylow of $\beta^{-1}(P_i')$ is a characteristic subgroup of $\beta^{-1}(P_i')$.
(5) Show that the p_i-Sylow of G is normal in G.

We have now shown that every Sylow subgroup of G is normal in G.

Problem B.5. Conclude by proving that G is the direct product of its Sylow subgroups.