D. The Schur-Zassenhaus theorem, general case

Today's goal is to prove:

> **Theorem (Schur-Zassenhaus):** Let A and C be finite groups with $\mathrm{GCD}(\#(A), \#(C)) = 1$. Then any short exact sequence $1 \to A \to B \to C \to 1$ is right split.

We introduce the following (not standard) terminology: We'll say that a pair of groups (A, C) is ***straightforward*** if every short exact sequence $1 \to A \to B \to C \to 1$ is right split. The abelian Schur-Zassenhaus theorem shows that if A is abelian and $\mathrm{GCD}(\#(A), \#(C)) = 1$, then (A, C) is straightforward.

Problem D.1. Suppose that (A_1, C) and (A_2, C) are straightforward and there is a short exact sequence $1 \to A_1 \to A \to A_2 \to 1$ with A_1 canonical in A. Show that (A, C) is straightforward. **Hint/Warning:** Unfortunately, I think this first problem is one of the hardest. First use that (A_2, C) is straightforward, then use that splitting to build a new sequence which we can split using that (A_1, C) is straightforward.

Problem D.2. Let C be a finite group, let p be a prime not dividing $\#(C)$ and let P be a p-group. Show that (P, C) is straightforward.

Let p be a prime dividing $\#(A)$ and let P be a p-Sylow subgroup of A. Let $1 \to A \to B \to C \to 1$ be a short exact sequence, with $\mathrm{GCD}(\#(A), \#(C)) = 1$. **Assume inductively that we have shown** (A', C) **is straightforward whenever** $\mathrm{GCD}(\#(A'), \#(C)) = 1$ **for** $\#(A') < \#(A)$**.**

Recall that $N_A(P) = \{a \in A : aPa^{-1} = P\}$ and likewise for $N_B(P)$.

Problem D.3. Show that P is canonical in $N_A(P)$.

Problem D.4. Suppose that $A = N_A(P)$. Prove that $1 \to A \to B \to C \to 1$ is right split.

So we may now assume that $N_A(P) \neq A$**.**

Problem D.5. With A, B, C, P as above, show that $1 \to N_A(P) \to N_B(P) \to C \to 1$ is exact.

Problem D.6. Show that $1 \to A \to B \to C \to 1$ is right split.