E. THE GALOIS CORRESPONDENCE

Recall:

Theorem/Definition Let L/K be a field extension of finite degree. The following are equivalent:

(1) We have $\# \operatorname{Aut}(L/K) = [L : K]$.
(2) The fixed field of $\operatorname{Aut}(L/K)$ is K.
(3) For every $\theta \in L$, the minimal polynomial of θ over K is separable and splits in L.
(4) L is the splitting field of a separable polynomial $f(x) \in K[x]$.

A field extension L/K which satisfies these equivalent definitions is called ***Galois***.

Given a subfield F with $K \subseteq F \subseteq L$, we write $\operatorname{Stab}(F)$ for the subgroup of G fixing F; given a subgroup H of $\operatorname{Gal}(L/K)$, we write $\operatorname{Fix}(H)$ for the subfield of L fixed by H. Our next main goal will be to show:

The fundamental Theorem of Galois theory Let L/K be a Galois extension with Galois group G. The maps Stab and Fix are inverse bijections between the set of subgroups of G and the set of intermediate fields F with $K \subseteq F \subseteq L$. Moreover, if $F_1 \subseteq F_2$, then $\operatorname{Stab}(F_1) \supseteq \operatorname{Stab}(F_2)$ and $[\operatorname{Stab}(F_1) : \operatorname{Stab}(F_2)] = [F_2 : F_1]$. If $H_1 \subseteq H_2$ then $\operatorname{Fix}(H_1) \supseteq \operatorname{Fix}(H_2)$ and $[\operatorname{Fix}(H_1) : \operatorname{Fix}(H_2)] = [H_2 : H_1]$.

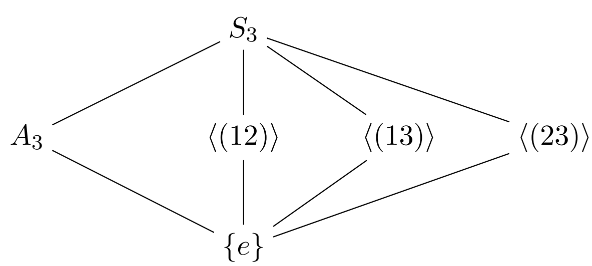
We start by proving some basic results about Fix and Stab.

Problem E.1. (1) Show that, if $F_1 \subseteq F_2$ then $\operatorname{Stab}(F_1) \supseteq \operatorname{Stab}(F_2)$.
(2) Show that, if $H_1 \subseteq H_2$ then $\operatorname{Fix}(H_1) \supseteq \operatorname{Fix}(H_2)$.

Problem E.2. (1) Show that $\operatorname{Stab}(\operatorname{Fix}(H)) \supseteq H$.
(2) Show that $\operatorname{Fix}(\operatorname{Stab}(F)) \supseteq F$.

The Fundamental Theorem tells us that both of the $\supseteq$'s in Problem E.2 are actually equality, but we don't know that yet.

We now give examples. Here is a table of the subgroups of S_3:



Problem E.3. Let $L = \mathbb{Q}(x_1, x_2, x_3)$, let S_3 act on L by permuting the variables and let $K = \operatorname{Fix}(S_3)$. Describe the subfield of L fixed by each of the subgroups of S_3.

Problem E.4. Let L be the splitting field of $x^3 - 2$ over $\mathbb{Q}$. We number the roots of $x^3 - 2$ as $\sqrt[3]{2}$, $\omega \sqrt[3]{2}$ and $\omega^2 \sqrt[3]{2}$, where ω is a primitive cube root of 1. Described the subfield of L fixed by each of the subgroups of S_3.

Now we prove the theorem!

Problem E.5. Both parts of this problem are things you already did, your job is just to remember when you did them.

(1) Let L/K be a Galois extension. Let F be a field with $K \subseteq F \subseteq L$. Show that $|\operatorname{Stab}(F)| = [L : F]$.
(2) Let L/K be a Galois extension. Let H be a subgroup of $\operatorname{Gal}(L/F)$. Show that $[L : \operatorname{Fix}(H)] = |H|$.

Problem E.6. Prove that the maps Fix and Stab in the Fundamental Theorem are mutually inverse.

Problem E.7. Check the remaining claims of the Fundamental Theorem.