PROBLEM SET ONE: DUE MIDNIGHT JANUARY 18

See the course website for homework policies. I have included several problems on group actions although we won't officially meet these definitions until Worksheets 4 and 5. This is material I expect you have seen before, but feel free to look at Worksheets 4 and 5 if you need reminders about these concepts.

Problem 1. Write up complete solutions to **three** of the following problems from the worksheets:

$$1.4, 2.2, 2.6, 2.7, 3.1, 3.4$$

Problem 2. Let C be the cube $[-1,1]^3$ in $\mathbb{R}^3$ and let G be the group of rotation and reflection symmetries of C. You needn't prove your answers to this problem correct.

(1) Describe the G-orbits of the points $(1,0,0)$, $(1,1,0)$, $(1,1,1)$.
(2) For each of the points x in the previous part, compute $\#\mathrm{Stab}(x)$.

Problem 3. Let S_4 act on the polynomial ring $\mathbb{C}[r_1,r_2,r_3,r_4]$ by permuting the variables. For each of the following polynomials, compute the size of its stabilizer. You needn't prove your answers to this problem correct.

$$f = r_1 \qquad g = r_1r_2 + r_3r_4 \qquad h = \prod_{1\le i<j\le 4}(r_i - r_j)$$

Problem 4. Let G be a group and let a be an element of G. Show that $g \mapsto aga^{-1}$ is a group homorphism $G \to G$.

Problem 5. In this problem, we will prove the ***fundamental theorem of symmetric functions***. Let k be a field, let R be the ring $k[r_1,r_2,\ldots,r_n]$ and let S be the ring of S_n invariant polynomials in R. For $1 \le d \le n$, define

$$e_d = \sum_{1\le i_1<i_2<\cdots<i_d\le n} r_{i_1}r_{i_2}\cdots r_{i_d}.$$

The fundamental theorem of symmetric functions states that S is generated (as a k-algebra) by e_1, e_2, ..., e_n. We'll write E for the sub-k-algebra of S generated by e_1, e_2, ..., e_n, so our goal is to show that $E = S$.

(1) Show that it is enough to show that, if $f \in S$ is homogenous of degree m, then $f \in E$.

Let M be the set of monomials $r_1^{a_1}r_2^{a_2}\cdots r_n^{a_n}$ of degree m. Place a total order on M by saying that $r_1^{a_1}r_2^{a_2}\cdots r_n^{a_n} > r_1^{b_1}r_2^{b_2}\cdots r_n^{b_n}$ if $a_1 = b_1$, $a_2 = b_2$, ..., $a_{j-1} = b_{j-1}$ and $a_j > b_j$ for some $1 \le j \le n$. For example, if $n = 3$ and $m = 2$, our order is

$$r_1^2 > r_1r_2 > r_1r_3 > r_2^2 > r_2r_3 > r_3^2.$$

We define the ***leading term*** of a nonzero homogenous degree m polynomial f to be the largest monomial of M whose coefficient is nonzero.

(2) Let f be a nonzero homogenous degree m polynomial in S and let $r_1^{a_1}r_2^{a_2}\cdots r_n^{a_n}$ be the leading term of f. Show that $a_1 \ge a_2 \ge \cdots \ge a_n$.
(3) Let $a_1 \ge a_2 \ge \cdots \ge a_n$ with $\sum a_j = m$. Show that there is a polynomial $g \in E$ with leading term $r_1^{a_1}r_2^{a_2}\cdots r_n^{a_n}$.
(4) Prove that $S = E$.