PROBLEM SET 10: DUE MIDNIGHT ON APRIL 5

See the course website for homework policies.

Problem 1. Remember to go to plan an hour to go to Gradescope and do Practice QR Exam 8.

Problem 2. Please write up the proofs of **three** of **20.2**, **20.3**, **20.4**, **21.5**, **21.7**, **21.8**.

Problem 3. This is a followup to problems 6 and 7 of Problem Set 9: Let n be a positive integer and let F be a field where $n \neq 0$. Let a be a nonzero element of F and let L be the splitting field of $x^n - a$. Show that $\mathrm{Gal}(L/F)$ is a subgroup of $(\mathbb{Z}/n\mathbb{Z}) \rtimes (\mathbb{Z}/n\mathbb{Z})^\times$.

Problem 4. Let p be a prime number and let $q = p^n$ be a power of p. Recall that we defined $\mathbb{F}_q$ to be the splitting field of $x^q - x$ over $\mathbb{F}_p$, and we showed that $\mathbb{F}_q$ is a field with q elements.

(1) Show that $\mathbb{F}_q/\mathbb{F}_p$ is Galois.
(2) We define the ***Frobenius map*** $\Phi : \mathbb{F}_q \to \mathbb{F}_q$ by $\Phi(x) = x^p$. Show that $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ is cyclic of order n, and is generated by Φ.

Problem 5. Let L/K be a field extension of degree n. For $\theta \in L$, let m_θ be the map $x \mapsto \theta x$ from L to L; this is clearly a K-linear map. Let $f(x) = x^d + f_{d-1}x^{d-1} + \cdots + f_0$ be the minimal polynomial of θ over K.

(1) Express the minimal polynomial and characteristic polynomial of m_θ in terms of $f(x)$, d and n.
(2) The ***trace*** $T_{L/K}(\theta)$ is defined to be the trace of the F-linear map m_θ. Express $T_{L/K}(\theta)$ in terms of the f_j, d and n.
(3) The ***norm*** $N_{L/K}(\theta)$ is defined to be the determinant of the linear map m_θ. Express $N_{L/K}(\theta)$ in terms of the f_j, d and n.

Now, suppose that L/K is a Galois extension.

(4) Show that $T_{L/K}(\theta) = \sum_{g \in \mathrm{Gal}(L/K)} g(\theta)$ and $N_{L/K}(\theta) = \prod_{g \in \mathrm{Gal}(L/K)} g(\theta)$.

Problem 6. This is a lemma we'll want soon, though it doesn't mention Galois theory:

(1) Let G be a subgroup of S_n. Define a relation $\sim$ on $\{1, 2, \ldots, n\}$ by saying that, for $1 \leq i \neq j \leq n$, we have $i \sim j$ if $(ij) \in G$ and defining $i \sim i$ for $1 \leq i \leq n$. Show that $\sim$ is an equivalence relation.
(2) Let p be prime, let G be a subgroup of S_p that acts transitively on $\{1, 2, \ldots p\}$ and suppose that G contains a transposition. Show that $G = S_p$.

Problem 7. Let L/K be a Galois extension, not of characteristic 2. Let $\{\theta_1, \theta_2, \ldots, \theta_N\}$ be a subset of L which is mapped to itself by $\mathrm{Gal}(L/K)$. Show that $K(\sqrt{\theta_1}, \ldots, \sqrt{\theta_N})/K$ is Galois.

Problem 8. Let L be a field, let G be a group of automorphisms of L, and let F be the fixed field of G. The group G acts on the vector space L^n by acting on each coordinate. Let V be a positive dimensional L-subspace of L^n such that $g(V) = V$ for all $g \in G$. In this problem, we will prove some useful lemmas about this situation. We write $(x_1, x_2, \ldots, x_n)$ for coordinates on L^n.

(1) Show that there is some integer $k \geq 0$ such that $V \cap \{x_1 = x_2 = \cdots = x_k = 0\}$ is one dimensional.
(2) With k as above, show that $V \cap \{x_1 = x_2 = \cdots = x_k = 0\}$ contains a nonzero vector with coefficients in F. In particular, V contains a nonzero vector with coefficients in F.
(3) Prove, more strongly, that V has a basis made of vectors with coefficients in F.