PROBLEM SET 11: DUE MIDNIGHT ON APRIL 12

Final problem set! See the course website for homework policies.

Problem 1. Remember to go to plan an hour to go to Gradescope and do Practice QR Exam 9.

Problem 2. Please write up the proofs of **two** of **22.3**, **22.4**, **22.5**.

Problem 3. Please write up the proof (assuming the background from the worksheets through worksheet 22) that the Galois correspondence is a bijection. To be precise: Let L/F be a Galois extension. For any field K with $F \subseteq K \subseteq L$, we define $\mathrm{Stab}(K) = \{g \in \mathrm{Gal}(L/F) : g(\theta) = \theta \ \forall \theta \in K\}$; for any subgroup H of $\mathrm{Gal}(L/F)$, we define $\mathrm{Fix}(H) = \{\theta \in L : g(\theta) = \theta \ \forall g \in H\}$. Show that Fix and Stab are mutually inverse bijections between $\{$fields between F and $L\}$ and $\{$subgroups H of $\mathrm{Gal}(L/F)$ $\}$.

Problem 4. Let L, F, Stab and Fix be as above.

(1) Let H_1 and H_2 be two subgroups of $\mathrm{Gal}(L/F)$. Show that $\mathrm{Fix}(H_1 \cap H_2)$ is the subfield of L generated by $\mathrm{Fix}(H_1)$ and $\mathrm{Fix}(H_2)$.
(2) Let K_1 and K_2 be two fields between F and L. Show that $\mathrm{Stab}(K_1 \cap K_2)$ is the subgroup of $\mathrm{Gal}(L/F)$ generated by $\mathrm{Stab}(K_1)$ and $\mathrm{Stab}(K_2)$.

Problem 5. Let $K = \mathbb{Q}(\sqrt{5+2\sqrt{5}})$. Show that $K/\mathbb{Q}$ is Galois and $\mathrm{Gal}(K/\mathbb{Q}) \cong C_4$. (Hint: $\sqrt{5-2\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5+2\sqrt{5}}}$.)

Problem 6. Let n be a positive integer and let F be a field where $n \neq 0$ and where $x^n - 1$ splits. Let $a \in F$ with $a \neq \pm 2$, and let K be the splitting field of the polynomial $x^{2n} - ax^n + 1$. Show that $\mathrm{Gal}(K/F)$ is a subgroup of the dihedral group of order $2n$.

Problem 7. Let F be a field of characteristic $\neq 2$. Let $f(x)$ be a separable polynomial in $F[x]$ and let L be a splitting field for f where f has roots $\{\theta_1, \theta_2, \ldots, \theta_n\}$. Put $\Delta = \prod_{i<j}(\theta_i - \theta_j)$.

(1) Show that Δ^2 is in F.
(2) Consider $\mathrm{Gal}(L/F)$ as a subgroup of S_n, acting on $\{\theta_1, \theta_2, \ldots, \theta_n\}$. Show that $\mathrm{Gal}(L/F) \subseteq A_n$ if and only if $\Delta \in F$.

Problem 8. Let F be a field, let $f(x)$ be an irreducible separable polynomial in $F[x]$ and let L be a splitting field for f where f has roots $\{\theta_1, \theta_2, \ldots, \theta_n\}$. We consider $\mathrm{Gal}(L/F)$ as a subgroup of S_n, acting on $\{\theta_1, \theta_2, \ldots, \theta_n\}$. Show that the following are equivalent:

(a) $\mathrm{Gal}(L/F)$ is the cyclic group $\langle (123 \cdots n) \rangle$
(b) There is a polynomial $p(x) \in F[x]$ such that $\theta_{i+1} = p(\theta_i)$ for $1 \leq i \leq n$ (with $\theta_{n+1} := \theta_1$.)

Problem 9. Let K/F be a field extension of finite degree for which there is a Galois extension L/F with $F \subseteq K \subseteq L$. (As we will learn in class, this always holds in characteristic zero.)

(1) Show that there are only finitely many fields K' with $F \subseteq K' \subseteq K$.
(2) (**The primitive element theorem**) Suppose that F is infinite. Show that there is an element $\theta \in K$ with $K = F(\theta)$.