PROBLEM SET TWO: DUE MIDNIGHT JANUARY 25

See the course website for homework policies.

Problem 1. Remember to go to plan an hour to go to Gradescope and do Practice QR Exam 1.

Problem 2. Write up complete solutions to **two** of the following problems from the worksheets:

$$4.4, 5.1, 5.5, 5.8$$

Problem 3. Let G be a group with n elements.

(1) Show that G is isomorphic to a subgroup of S_n.
(2) Let k be a field. Show that G is isomorphic to a subgroup of $\mathrm{GL}_n(k)$.

Problem 4. Let p be a prime number and let G be a group of order p. Show that G is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

Problem 5. Prove ***Burnside's Lemma:*** Let G be a finite group and let X be a finite set on G acts. Show that

$$\frac{1}{\#G} \sum_{g \in G} \#\mathrm{Fix}(g) = \#(G \backslash X).$$

Problem 6. Let V be a vector space over some field. Let G be the set $V \times \bigwedge^2 V$ and define a multiplication operation on G by $(v, \alpha) * (w, \beta) = (v + w, \alpha + \beta + v \wedge w)$. Check that G is a group.

Problem 7. Let G be a group and let $g \in G$. The ***conjugacy class*** of g, written $\mathrm{Conj}(g)$, is $\{hgh^{-1} : h \in G\}$ and the ***centralizer*** of g, written $Z(g)$, is $\{h \in G : gh = hg\}$. Suppose that G is finite.

(1) Show that $\#(G) = \#\mathrm{Conj}(g) \cdot \#Z(g)$.
(2) Let $g_1, g_2, \ldots, g_c$ be one representative from each conjugacy class of G. Prove the ***class equation*** $\sum_i \frac{1}{\#Z(g_i)} = 1$.

Problem 8. Let p be a prime and let G be a group of order p^k for some $k \geq 1$.

(1) Show that there is some $g \in G$ other than e such that $\mathrm{Conj}(g) = \{g\}$.
(2) Show that this g commutes with every $h \in G$.

Problem 9. Let G be a finite group with 3 conjugacy classes. Show that $\#(G) \leq 6$. Hint: See Problem 7.