PROBLEM SET THREE: DUE MIDNIGHT FEBRUARY 1

See the course website for homework policies.

Problem 1. Remember to go to plan an hour to go to Gradescope and do Practice QR Exam 2.

Problem 2. This problem is your opportunity to write up and improve Problem 6.1 from the worksheets: Let G be a group and let N be a subgroup of G. Show that the following are equivalent:

(1) For all $g \in G$, we see $N = gNg^{-1}$.
(2) N is a union of (some of the) conjugacy classes of G.
(3) All elements of G/N have the same stabilizer, for the left action of G on G/N.
(4) For all $g \in G$, we have $gN = Ng$.
(5) For all $g \in G$, there is a $g' \in G$ such that $gN = Ng'$.
(6) For all g_1 and $g_2 \in G$, we have $g_1 N g_2 N = g_1 g_2 N$.
(7) There is a group structure on G/N such that the map $g \mapsto gN$ from G to G/N is a group homomorphism.
(8) There is a group H and a group homomorphism $\alpha : G \to H$ such that $N = \mathrm{Ker}(\alpha)$.

Many of the implications are very short. That's fine!

Problem 3. Write up worksheet problem **6.3**, **6.4**, **7.1** or **7.2**.

Problem 4. This problem is meant to help you think about conjugacy classes: Let G be a group and let g_1 and g_2 be conjugate elements of G.

(1) Show that g_1 and g_2 have the same order.
(2) For any positive integer k, show that g_1 has a k-th root in G if and only if g_2 has a k-th root.
(3) Suppose that G acts on a set X, and let $X_i = \{x \in X : g_i(x) = x\}$. Show that there is an $h \in G$ such that $hX_1 = X_2$.

Problem 5. Let G be a group, N a normal subgroup of G and g an element of N. Let $\mathrm{Conj}_G(g)$ be the conjugacy class of g in G. In this problem, we will discuss how $\mathrm{Conj}_G(g)$ splits into conjugacy classes of N.

(1) Show that $\mathrm{Conj}_G(g)$ is a union of conjugacy classes of N.
(2) Suppose that G/N is finite. Let $C_G(g) = \{h \in G : gh = hg\}$. Let π be the quotient map $G \to G/N$. Show the number of N-conjugacy classes in $\mathrm{Conj}_G(g)$ is $\frac{\#(G/N)}{\#\pi(C_G(g))}$.
(3) Let n be an odd number. Show that S_n-conjugacy class of $(123 \cdots n)$ splits into two conjugacy classes in A_n.
(4) Let p be an odd prime. Show that the $\mathrm{GL}_2(\mathbb{F}_p)$-conjugacy class of $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ splits into two conjugacy classes in $\mathrm{SL}_2(\mathbb{F}_p)$.

Problem 6. Let G be a group and H a subgroup. The subgroup H is called ***canonical*** if $\phi(H) = H$ for every automorphism ϕ of G.

(1) Show that, if H is canonical in G, then H is normal in G.

Let $A \subset B \subset C$ be a chain of groups. Give a proof or counterexample to each statement:

(2) If A is canonical in B and B is canonical in C, then A is canonical in C.
(3) If A is canonical in B and B is normal in C, then A is normal in C.
(4) If A is normal in B and B is canonical in C, then A is normal in C.
(5) If A is normal in B and B is normal in C, then A is normal in C.