PROBLEM SET FOUR: DUE MIDNIGHT FEBRUARY 8

See the course website for homework policies.

Problem 1. Remember to go to plan an hour to go to Gradescope and do Practice QR Exam 3.

Problem 2. Write up two of worksheet problems **8.2**, **8.5** and **8.6**.

Problem 3. The ***quaternion eight group***, Q_8, is the subgroup $\{\pm 1, \pm i, \pm j, \pm k\}$ of the quaternions. Recall that $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. Show that there are short exact sequences $1 \to C_2 \to Q_8 \to C_2^2 \to 1$ and $1 \to C_4 \to Q_8 \to C_2 \to 1$.

Problem 4. Let U_+ and B_+ be the subgroups of $\mathrm{GL}_n(K)$ consisting of matrices of the form

$$U_+ = \left\{ \begin{bmatrix} 1 & * & \cdots & * \\ & 1 & \cdots & * \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} \right\} \subset \mathrm{GL}_n(K) \qquad B_+ \left\{ \begin{bmatrix} * & * & \cdots & * \\ & * & \cdots & * \\ & & \ddots & \vdots \\ & & & * \end{bmatrix} \right\} \subseteq \mathrm{GL}_n(K)$$

Show that there is a short exact sequence $1 \to U_+ \to B_+ \to (K^\times)^n \to 1$.

Problem 5. Show that there is a short exact sequence

$$0 \to \mathbb{Z} \to \begin{bmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{bmatrix} \to \mathbb{Z}^2 \to 0.$$

Problem 6. Let G be a group; let A and B be subgroups of G. Define $AB = \{ab : a \in A,\ b \in B\}$.

(1) Give an example of a group G and subgroups A and B where AB is not a subgroup of G.
(2) Suppose that $A \cap B = \{e\}$. Show that each element of AB can be written as ab, for $a \in A$ and $b \in B$, in only one way.
(3) Suppose that, for all $a \in A$, we have $aBa^{-1} = B$. Show that AB is a subgroup of G.
(4) Suppose that $A \cap B = \{e\}$, that $aBa^{-1} = B$ for $a \in A$ and that $bAb^{-1} = A$ for $b \in B$. Show that $AB \cong A \times B$.

Problem 7. In this problem, we will prove the "snake lemma for groups". Let

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A_1 & \xrightarrow{\iota_1} & B_1 & \xrightarrow{\pi_1} & C_1 & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 1 & \longrightarrow & A_2 & \xrightarrow{\iota_2} & B_2 & \xrightarrow{\pi_2} & C_2 & \longrightarrow & 1 \end{array}$$

be a commutative diagram of groups, where the rows are exact. We define a map $\delta : \mathrm{Ker}(\gamma) \to A_2/\alpha(A_1)$ as follows: For $c_1 \in \mathrm{Ker}(\Gamma)$, lift c_1 to an element b_1 in B_1. Note that $\beta(b_1)$ lies in $\iota_2(A_2)$. The map δ will send c_1 to the coset $\iota_2^{-1}(\beta(b_1))\alpha(A_1)$.

(1) Show that the coset the coset $\iota_2^{-1}(\beta(b_1))\alpha(A_1)$ is independent of the choice of b_1, so δ is well defined.

So we have a sequence of maps

$$\{1\} \to \mathrm{Ker}(\alpha) \to \mathrm{Ker}(\beta) \to \mathrm{Ker}(\gamma) \xrightarrow{\delta} A_2/\alpha(A_1) \to B_2/\beta(B_1) \to C_2/\gamma(C_1) \to \{1\}. \qquad (*)$$

Check that:

(2) For c_1 in $\mathrm{Ker}(\gamma)$, check that $\delta(c_1) = e\alpha(A_1)$ if and only if $c_1 \in \pi_1(\mathrm{Ker}(\beta))$.
(3) For a coset $a_2\alpha(A_1)$, show that $\iota_2(a_2)\beta(B_1) = e\beta(B_1)$ if and only if $a_2\alpha(A_1) \in \delta(\mathrm{Ker}(\gamma))$.

The Snake Lemma says that, on each segment $X \to Y \to Z$ of $(*)$, the image of X is precisely the preimage of the identity element/coset of Z. We'll stop here, rather than checking the remaining four cases. Also, I won't assign this, but if $\alpha(A_1) \trianglelefteq A_2$, then δ is a group homomorphism.