PROBLEM SET FIVE: DUE MIDNIGHT ~~FEBRUARY 15~~ FEBRUARY 16

See the course website for homework policies.

Problem 1. No new quiz this week. Instead, write up your choice of two problems from the first three quizzes. You can find the old quizzes at
`http://www.math.lsa.umich.edu/~speyer/594/CombinedQuizzes.pdf`

Problem 2. Write up **three** of worksheet problems **9.1**, **9.4**, **9.6**, **9.7**, **10.2**, **10.3**, **10.4**, **10.9**.

Problem 3. Recall that, for a finite group G, the length $\ell(G)$ is the number of quotients in any composition series of G. Give an example of finite groups $H \subset G$ with $\ell(H) > \ell(G)$. (Hint: What is the length of a symmetric group?)

Problem 4. Let G be the group of bijections $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ of the form $x \mapsto ax + b$, with $a \in (\mathbb{Z}/n\mathbb{Z})^\times$ and $b \in \mathbb{Z}/n\mathbb{Z}$.

(1) Show that G is solvable.
(2) If n is odd, show that $G^{\mathrm{ab}} \cong (\mathbb{Z}/n\mathbb{Z})^\times$.
(3) Can you describe G^{ab} if n is even?

Problem 5. Let k be a field and let B be the group of invertible upper triangular $n \times n$ matrices with entries in k. Show that B is solvable.

Problem 6. Let V be a vector space over a field of characteristic $\neq 2$. Let G be the group[1] from Problem Set 2, Problem 6: $G = V \times \bigwedge^2 V$ as a set and $(v, \alpha) * (w, \beta) = (v + w, \alpha + \beta + v \wedge w)$.

(1) Give a criterion for an element $(w, \gamma) \in G$ to be expressible as a commutator: $(w, \gamma) = (u, \alpha)(v, \beta)(u, \alpha)^{-1}(v, \beta)^{-1}$.
(2) Show that the commutator subgroup of G is $\{0\} \times \bigwedge^2 V$.
(3) Assuming that $\dim V \geq 4$, show that not every element of the commutator subgroup is a commutator.

Problem 7. Let F be a field. Let $\mathrm{GL}_n(F)$ be the group of invertible $n \times n$ matrices with entries in F and let $\mathrm{SL}_n(F)$ be the group of submatrices with determinant 1. The aim of this problem is to describe the abelianization of $\mathrm{GL}_n(F)$ and $\mathrm{SL}_n(F)$.

For $1 \leq i \neq j \leq n$ and $r \in F$, we define $E_{ij}(r)$ to be the matrix with ones on the diagonal, an r in position (i, j) and zeroes everywhere else; we call such a matrix an ***elementary matrix***. We showed in Math 593 (and you may use) that the elementary matrices generate $\mathrm{SL}_n(F)$.

(1) Suppose that $n \geq 3$. Show that the elementary matrices are in the commutator subgroup of $\mathrm{SL}_n(F)$. Conclude that the abelianization of $\mathrm{SL}_n(F)$ is trivial and the abelianization of $\mathrm{GL}_n(F)$ is $F^\times$. Hint: First think about matrices of the form $\left[\begin{smallmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{smallmatrix}\right]$.
(2) Show that the elementary matrices are in the commutator subgroup of $\mathrm{GL}_2(F)$ for $\#(F) > 2$ and in the commutator subgroup of $\mathrm{SL}_2(F)$ for $\#(F) > 3$. Conclude that the abelianization of $\mathrm{SL}_2(F)$ is trivial and the abelianization of $\mathrm{GL}_2(F)$ is $F^\times$ in these cases. Hint: First think about matrices of the form $\left[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right]$.
(3) What are the abelianizations of $\mathrm{GL}_2(\mathbb{F}_2) = \mathrm{SL}_2(\mathbb{F}_2)$ and $\mathrm{SL}_2(\mathbb{F}_3)$?

---

[1]If you are wondering how to see intuitively that G is a group: Let $\bigwedge^\bullet(V)$ be the exterior algebra of V, and let $\langle \bigwedge^3(V) \rangle$ be the two sided ideal of $\bigwedge^\bullet(V)$ generated by $\bigwedge^3(V)$. Then $\bigwedge^\bullet(V)/\langle \bigwedge^3(V) \rangle$ is a ring, and thus has a unit group. The group G is a subgroup of the unit group, made of elements of the form $1 + v + \alpha$ for $v \in V$ and $\alpha \in \bigwedge^2(V)$.