PROBLEM SET SIX: DUE MIDNIGHT FEBRUARY 22

See the course website for homework policies.

Problem 1. Remember to go to plan an hour to go to Gradescope and do Practice QR Exam 4. Both problems will relate to solvable groups.

Problem 2. Please write up **two** of **11.3, 11.5, 12.1, 12.3, 12.7**

Problem 3. Which of the following sequences are left split? Which are right split? Which are neither?

(1) $1 \to C_2 \to C_6 \to C_3 \to 1$.
(2) $1 \to C_2 \to C_4 \to C_2 \to 1$.
(3) $1 \to A_5 \to S_5 \xrightarrow{\epsilon} \{\pm 1\} \to 1$.
(4) $1 \to \mathrm{SL}_4(\mathbb{R}) \to \mathrm{GL}_4(\mathbb{R}) \xrightarrow{\det} \mathbb{R}^\times \to 1$.

Problem 4. Let $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ be a short exact sequence of groups. Let $\tilde{C}$ be a subset of B such that $\beta : \tilde{C} \to C$ is bijective. Show that each element of B can be uniquely written as $\alpha(a)\tilde{c}$ for $a \in A$ and $\tilde{c} \in \tilde{C}$.

Problem 5. Let $1 \to A \to B \to C \to 1$ be a short exact sequence of groups. Let B act on a set X in such a way that the action of A on X has exactly one orbit and trivial stabilizers. Show that the sequence $1 \to A \to B \to C \to 1$ is right split.

Definition: Let G be a group. A ***central series*** of G is a sequence of subgroups $G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_N$ such that, if $g \in G$ and $h \in G_i$ then $ghg^{-1}h^{-1} \in G_{i-1}$, for $1 \leq i \leq N$. G is called ***nilpotent*** if it has a central series $G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_N$ with $G_0 = \{e\}$ and $G_N = G$.

Problem 6. Let k be a field and let U be the group of matrices with entries in k of the form

$$\begin{bmatrix} 1 & * & * & \cdots & * \\ & 1 & * & \cdots & * \\ & & 1 & \cdots & * \\ & & & \ddots & \vdots \\ & & & & 1 \end{bmatrix}.$$

Show that U is nilpotent.

Problem 7. (1) Let G be a nontrivial nilpotent group. Show that G has a nontrivial center.
(2) Show that S_3 is not nilpotent (although it is solvable).
(3) Let F be a field with more than two elements. Show that the group of invertible matrices of the form $\left[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right]$ with entries in F is not nilpotent (although it is solvable).

Problem 8. Let G be a nilpotent group.

(1) Show that every subgroup of G is nilpotent.
(2) Show that every quotient of G is nilpotent.

Problem 9. (1) Let $1 \to Z \xrightarrow{\alpha} G \to H \to 1$ be a short exact sequence where $\alpha(Z)$ is central in G and H is nilpotent. Show that G is nilpotent.
(2) Give an example of a short exact sequence $1 \to A \to G \to H \to 1$ where A is abelan and H is nilpotent, but G is not nilpotent.

Problem 10. Let p be a prime and let G be a group of size p^k. Show that G is nilpotent. (Hint: A previous homework problem will help.)