PROBLEM SET SEVEN: DUE MIDNIGHT ON MARCH 15

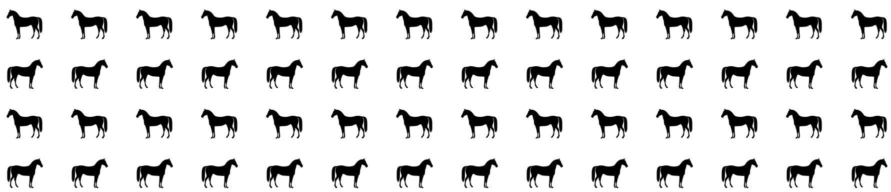
See the course website for homework policies.

Problem 1. Remember to go to plan an hour to go to Gradescope and do Practice QR Exam 5.

Problem 2. Please write up **two** of **13.4, 13.5, 13.6, 13.7**.

Problem 3. Describe all actions of C_2 on C_{15} by group automorphisms.

Problem 4. Let G be the group of all symmetries of the plane of the form $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} (-1)^b x+a \\ y+b \end{smallmatrix}\right]$ for $a, b \in \mathbb{Z}$. This is the group of symmetries of the pattern below (imagined to fill the plane).



(1) Show that there is a short exact sequence $1 \to \mathbb{Z}^2 \to G \to C_2 \to 1$.
(2) Show that this sequence is not right split.

Problem 5. Let R be a ring (not assumed commutative) and let I be a two sided ideal of R. We define I^m to be the two sided ideal generated by all products $x_1 x_2 \cdots x_m$ for $x_1, x_2, \ldots, x_m \in I$. We define the ideal N to be ***nilpotent*** if there is a positive integer m such that $N^m = (0)$. Let N be a nilpotent ideal and let U be the group $\{1 + x : x \in N\}$. Show that U is a nilpotent group.

Problem 6. Let G be a group.

(1) The ***upper (or ascending) central series*** of G is defined inductively as follows: $U_0 = \{e\}$ and $U_{k+1} = \pi_k^{-1}(Z(G/U_k))$, where π_k is the projection $G \to G/U_k$. Show that G is nilpotent if and only if $U_N = G$ for some G.
(2) The ***lower (or descending) central series*** is defined inductively as follows: $L^0 = G$ and L^{k+1} is the group generated by all products $ghg^{-1}h^{-1}$ with $g \in G$ and $h \in L^k$. Show that G is nilpotent if and only if $L^N = \{e\}$ for some G.

Problem 7. Let p be an odd prime and let G be a group of order p^3. The aim of this problem is to show that G is isomorphic to one of:

$$C_p^3, \quad C_{p^2} \times C_p, \quad C_{p^3}, \quad C_p^2 \rtimes_{k \mapsto \left[\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right]} C_p, \quad C_{p^2} \rtimes_{k \mapsto (1+kp)} C_p, .$$

(1) Show that there is a central extension $1 \to Z \to G \to C \to 1$ with $Z \cong C_p$ and either $C \cong C_p^2$ or $C \cong C_{p^2}$.
(2) If $C \cong C_{p^2}$, show G is abelian and in the list above. **From now on, we assume** $C \cong C_p^2$**.**
(3) Let g_1 and $g_2 \in G$ and set $z = g_1 g_2 g_1^{-1} g_2^{-1}$. Show that g_1^p, g_2^p and z are in Z.
(4) Show that $(g_1 g_2)^k = g_1^k g_2^k z^{-\binom{k}{2}}$.
(5) Show that the map $g \mapsto g^p$ is a group homomorphism $G \to Z$ and that it factors through the quotient C. (Here is where you will need that p is odd; this is false for Q_8.)
(6) Show that we can choose g_1 and g_2 in G, mapping to a basis of C, such that $g_1^p = 1$.
(7) Set $A' = \langle g_2 \rangle Z$ and $C' = \langle g_1 \rangle$. Show that $G = A' \rtimes C'$ and that A' is either C_p^2 or C_{p^2}.
(8) Classify the actions of C' on A' and thus show that G is one of the groups listed above.