PROBLEM SET EIGHT: DUE MIDNIGHT ON MARCH 22

See the course website for homework policies.

Problem 1. Remember to go to plan an hour to go to Gradescope and do Practice QR Exam 6. Both questions will focus on the Sylow theorems.

Problem 2. Please write up the proofs of three of **14.3**, **14.5**, **14.9**, **15.4**, **15.6**, **15.7**.

Problem 3. On Worksheet 15, you checked that there are no non-abelian simple groups of order < 60. Explain why this means that every group of order < 60 is solvable.

Problem 4. Let n be a positive integer. Show that a group of order $8 \cdot 7^n$ must be solvable.

Problem 5. The point of this problem is to give a direct proof that symmetric groups have p-Sylow subgroups. For any positive integer M, let $v_p(M)$ be the exponent of p in the prime factorization of M. Let n be a positive integer and write $n = pm + r$ for $0 \leq r < p - 1$.

(1) Show that $v_p(n!) = m + v_p(m!)$.
(2) Let P be a p-Sylow subgroup of S_m. Show that S_n has a subgroup Q of the form $C_p^m \rtimes P$ and that Q is a p-Sylow subgroup of S_n.

Problem 6. The goal of this problem is to prove the following theorem: A finite group G is nilpotent if and only if it is a direct product of p-groups.

(1) Prove the easy direction: A direct product of p-groups is nilpotent.

Now, let G be a finite nilpotent group of size $n = \prod_p p^{a_p}$. Suppose, inductively, that we know that all nilpotent groups of size $< n$ are products of p-groups.

(2) Show that there is a central extension $1 \to C_q \to G \xrightarrow{\beta} G' \to 1$ where q is prime and $G' = \prod_p P'_p$ where each P'_p is a p-group. (This is a product where each prime p appears once, so different factors are p-groups for different p's.)
(3) Set $P_q = \beta^{-1}(P'_q)$. Show that P_q is normal in G.

Now let p be a prime not equal to q.

(4) Show that $\beta^{-1}(P'_p) \cong C_q \times P'_p$. (Hint: abelian Schur-Zassenhaus theorem.)
(5) Let P_p be a p-Sylow subgroup of $\beta^{-1}(P'_p)$. Show that P_p is characteristic in $\beta^{-1}(G'_p)$.
(6) Show that P_p above is normal in G.
(7) Show that $G = \prod_p P_p$. (The product is over all primes, both $p = q$ and $p \neq q$.)

Let k be a field and let $k[x]$ be the ring of polynomials with coefficients in x. Recall that $k[x]$ is a PID; you may use this fact freely in these problems.

Problem 7. Let $K \subset L$ be two fields and let $a(x)$ and $b(x) \in K[x]$. Let $g(x)$ be the GCD of a and b in $K[x]$. Show that g is also the GCD of a and b in $L[x]$.

Problem 8. For a polynomial $f(x) = \sum f_j x^j \in k[x]$, we define $f'(x)$ to be $\sum j f_j x^{j-1}$.

(1) For any two polynomials $f(x)$ and $g(x) \in k[x]$, show that $(f+g)'(x) = f'(x) + g'(x)$ and $(fg)'(x) = f(x)g'(x) + f'(x)g(x)$. Note that your proof should work for any field k.

For a nonzero polynomial $f(x)$ and an irreducible polynomial $p(x)$, let $m_p(f)$ be the number of times that p appears in the prime factorization of f.

(2) Let f and p be as above and suppose that $m_p(f) > 0$. Show that, if k has characteristic zero, then $m_p(f') = m_p(f) - 1$.
(3) Give an example to show that the above need not be true in nonzero characteristic.