The quadratic formula

Let the quadratic $x^2 - e_1x + e_2$ factor as

$$x^2 - e_1x + e_2 = (x - r_1)(x - r_2).$$

Set $D = r_1 - r_2$ and note that $e_1 = r_1 + r_2$, so

$$r_1 = \tfrac{1}{2}(e_1 + D).(\clubsuit)$$

Now note that

$$D^2 = (r_1 - r_2)^2 = r_1^2 - 2r_1r_2 + r_2^2 = (r_1 + r_2)^2 - 4r_1r_2 = e_1^2 - 4e_2. \qquad (\heartsuit)$$

So, use $(\heartsuit)$ to compute D^2, take a square root to compute D, and use $(\clubsuit)$ to compute r_1.

The cubic formula

Let the cubic $x^3 - e_1x^2 + e_2x - e_3$ factor as

$$x^3 - e_1x^2 + e_2x - e_3 = (x - r_1)(x - r_2)(x - r_3).$$

Let $\omega = \frac{-1+\sqrt{-3}}{2}$. We note that $\omega^2 + \omega + 1 = 0$ and $\omega^3 = 1$. Put

$$P = r_1 + \omega r_2 + \omega^2 r_3 \qquad Q = r_1 + \omega^2 r_2 + \omega r_3.$$

We also note that $e_1 = r_1 + r_2 + r_3$. Thus, if we can compute P and Q, we can compute

$$r_1 = \frac{1}{3}(e_1 + P + Q). \qquad (\clubsuit_3)$$

Consider the quadratic equation with roots P^3 and Q^3:

$$y^2 - f_3y + f_6 = (y - P^3)(y - Q^3). \qquad (\diamondsuit_3)$$

We have

$$f_3 = 2e_1^3 - 9e_1e_2 + 27e_3 \qquad f_6 = (e_1^2 - 3e_2)^3. \qquad (\heartsuit_3)$$

Thus, we can use $(\heartsuit_3)$ to compute f_3 and f_6; use the quadratic formula to solve $(\diamondsuit_3)$ for P^3 and Q^3; take cube roots to get P and Q and then, finally, use $(\clubsuit_3)$ to find r_1. This is the cubic formula!

Technical note: The summary here suggests that you should take cube roots twice. This would give you 9 options for (P, Q), only 3 of which lead to correct solutions. In fact, you should compute one cube root to find P, and then compute Q as $\frac{e_1^2 - 3e_2}{P}$.

The quartic formula

Let the quartic $x^4 - e_1x^3 + e_2x^2 - e_3x + e_4$ factor as

$$x^4 - e_1x^3 + e_2x^2 - e_3x + e_4 = (x - r_1)(x - r_2)(x - r_3)(x - r_4).$$

Put

$$T = r_1 + r_2 - r_3 - r_4 \qquad U = r_1 - r_2 + r_3 - r_4 \qquad V = r_1 - r_2 - r_3 + r_4$$

We also note that $e_1 = r_1 + r_2 + r_3 + r_4$. Thus, if we can compute T, U and V, then we can compute

$$r_1 = \frac{1}{4}(e_1 + T + U + V). \qquad (\clubsuit_4)$$

Consider the cubic equations with roots T^2, U^2 and V^2:

$$x^3 - g_2x^2 + g_4x - g_6 = (x - T^2)(x - U^2)(x - V^2). \qquad (\diamondsuit_4).$$

We have

$$g_2 = 3e_1^2 - 8e_2 \qquad g_4 = 3e_1^4 - 16e_1^2e_2 + 16e_2^2 + 16e_1e_3 - 64e_4 \qquad g_6 = (e_1^3 - 4e_1e_2 + 8e_3)^2. \qquad (\heartsuit_4)$$

Thus, we can use $(\heartsuit_4)$ to compute g_2, g_4 and g_6; use the cubic formula equation to solve $(\diamondsuit_4)$ for T^2, U^2 and V^2; take square roots to get T, U and V and then, finally, use $(\clubsuit_4)$ to find r_1. This is the quartic formula!

Technical note: The summary here suggests that you should take square roots three times. This would give you 8 options for (T, U, V), only 4 of which lead to correct solutions. In fact, you should compute square roots to find T and U and then compute V as $\frac{e_1^3 - 4e_1e_2 + 8e_3}{TU}$.