THE JORDAN-HOLDER THEOREM – NOTES FOR WEDNESDAY, JANUARY 23

Let G be a finite group. Suppose we have two chains of subgroups

$$0 = G_0 \subsetneq G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_r = G$$

and

$$0 = H_0 \subsetneq H_1 \subsetneq H_2 \subsetneq \cdots \subsetneq H_s = G$$

so that G_i is normal in G_{i+1}, with G_{i+1}/G_i simple and H_j is normal in H_{j+1}, with H_{j+1}/H_j simple.

Theorem (Jordan-Holder): In the above setting, we have $r = s$, and the list of quotients $(G_1/G_0, G_2/G_1, \ldots, G_r/G_{r-1})$ is a rearrangement of the list of quotients $(H_1/H_0, H_2/H_1, \ldots, H_s/H_{s-1})$.

The set up of the theorem is easy to achieve:

Theorem 2: Let G be any finite group. Then we can find a chain of normal subgroups

$$0 = G_0 \subsetneq G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_r = G$$

so that G_i is normal in G_{i+1} and G_{i+1}/G_i is simple.

Proof: Let r be the largest possible number so that there is a chain

$$0 = G_0 \subsetneq G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_r = G$$

with G_i is normal in G_{i+1}. We can always take $r = 1$, and $(G_0, G_1) = (\{1\}, G)$, so such an r exists and, since G is finite, there is a largest such r.

We claim that G_{i+1}/G_i is simple. If not, let H be a normal subgroup of G_i/G_{i+1} and let π be the projection map $G_i \to G_i/G_{i+1}$. Then $\pi^{-1}(H)$ is normal in G_{i+1}, and G_i is normal in H. So

$$0 = G_0 \subsetneq G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_i \subsetneq \pi^{-1}(H) \subsetneq G_{i+1} \subsetneq \cdots \subsetneq G_r = G$$

is a longer chain, contradiction. $\square$

Such a chain is called a ***composition series*** for G.

Preliminary lemmas

Lemma 1: Let G be a group, H a normal subgroup and A an arbitrary subgroup of G. Then AH is a subgroup of G, where AH is the set of products $\{ah : \ a \in A, \ h \in H\}$.

Proof 1: We need to show that AH is closed under multiplication and inverses. For the first, let a_1, $a_2 \in A$ and h_1, $h_2 \in H$. Notice that

$$(a_1 h_1)(a_2 h_2) = (a_1 a_2)(a_2^{-1} h_1 a_2 h_2).$$

Since H is normal, we have $a_2^{-1} h_1 a_2 \in H$. So $(a_2^{-1} h_1 a_2 h_2) \in H$ and, clearly, $a_1 a_2 \in A$.

Similarly, for $a \in A$ and $h \in H$, we have

$$(ah)^{-1} = h^{-1} a^{-1} = a^{-1}(a h^{-1} a^{-1})$$

and we see that $(ah)^{-1} \in AH$. $\square$

Proof 2: Let π be the map $G \to G/H$. Then $AH = \pi^{-1}(A)$, which is clearly a subgroup of G. $\square$

Lemma 2 (Theorem 9 in the textbook): Let G be a group, with A and B normal subgroups and $A \cap B = \{1_G\}$. Then $A \times B \cong AB$, and the isomorphism is $(a, b) \mapsto ab$.

This was done in class on Jan. 14; we repeat the proof:

Proof: We want to show that $(a_1b_1)(a_2b_2) = (a_1a_2)(b_1b_2)$. So we must show that $b_1a_2 = a_2b_1$. In other words, we must check that, for $a \in A$ and $b \in B$, we have $ab = ba$.

Set $c = aba^{-1}b^{-1}$. Then $c = (aba^{-1})b^{-1}$ is in B. But, since $c = a(ba^{-1}b^{-1})$, we also have $a \in A$. So $c \in A \cap B$, and we have $c = 1_G$. We have shown $aba^{-1}b^{-1} = 1$; rearranging gives $ab = ba$.$\square$

Lemma 3: Let G be a group, let M and N be normal subgroups, $M \neq N$, with G/M and G/N simple. Then $G/(M \cap N) = G/M \times G/N$ and we have $M/(M \cap N) \cong G/N$ and $N/(M \cap N) \cong G/M$.

Proof: Notice that we cannot have $M \subsetneq N$. If we did, then N/M is normal in G/M and is neither G/M nor $\{1\}$, contradicting that G/M is simple. So $M \not\subseteq N$. Similarly, $N \not\subseteq M$.

Set $K = M \cap N$. Since it is an intersection of two normal subgroups, it is normal. Replace (G, M, N) by $(G/K, M/K, N/K)$. This preserves the truth of all the hypotheses and all the conclusions, and gives us the additional tool of letting us assume that $M \cap N = \{1\}$.

From the preceding lemma, we have $MN = M \times N$. Also, since M and N are normal, the subgroup MN is normal in G. We claim that $G = MN$. **Proof:** Since MN is normal in G, the subgroup MN/M is normal in G/M. Since G/M is simple, we have $MN/N = G/M$ or $MN/M = \{1\}$. The second possibility is wrong because $N \not\subseteq M$, so $MN \neq M$. The first possibility, must then be right: $MN/M = G/M$. So $MN = G$. We have deduced that $G \cong M \times N$. That specific statement is only right with the additional tool of $M \cap N = \{1\}$. But the consequence $G/(M \cap N) \cong G/M \times G/N$ lifts back to the original case, as do the others. $\square$

The Jordan-Holder theorem

Our proof is by induction on $|G|$. The base case, $|G| = 1$, is trivial. We use the symbol $\sim$ to denote "are rearrangements of each other".

Fix two chains $0 = G_0 \subsetneq G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_r = G$ and $0 = H_0 \subsetneq H_1 \subsetneq H_2 \subsetneq \cdots \subsetneq H_s = G$ as in the statement of the theorem. If $G_{r-1} = H_{s-1}$ then we are done by induction, applying the inductive hypothesis to the group G_{r-1}. So we may assume that $G_{r-1} \neq H_{s-1}$. We set

$$M = G_{r-1} \quad N = H_{s-1} \quad K = M \cap N.$$

Using Lemma 2, we can find a composition series

$$0 = K_0 \subsetneq K_1 \subsetneq K_2 \subsetneq \cdots \subsetneq K_t = K$$

for K.

Then $0 = G_0 \subsetneq G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_{r-1} = M$ and $0 = K_0 \subsetneq K_1 \subsetneq K_2 \subsetneq \cdots \subsetneq K_t = K \subsetneq M$ are both composition series for M. By induction, we have $r - 1 = t + 1$ and we have

$$(G_1/G_0, G_2/G_1, \ldots, G_{r-2}/G_{r-1}) \sim (K_1/K_0, K_2/K_1, \ldots, K_t/K_{t-1}, M/K). \quad (*)$$

Similarly, $0 = H_0 \subsetneq H_1 \subsetneq H_2 \subsetneq \cdots \subsetneq H_{s-1} = N$ and $0 = K_0 \subsetneq K_1 \subsetneq K_2 \subsetneq \cdots \subsetneq K_t = K \subsetneq N$ are both composition series for N. So $s - 1 = t + 1$ and the successive quotients

$$(H_1/H_0, H_2/H_1, \ldots, H_{s-2}/H_{s-1}) \sim (K_1/K_0, K_2/K_1, \ldots, K_t/K_{t-1}, N/K). \quad (**)$$

From the equalities $r-1 = t+1 = s-1$ we immediately deduce $r = s$. Taking $(*)$ and appending the quotient G/M to both lists, we have

$$(G_1/G_0, G_2/G_1, \ldots, G_{r-2}/G_{r-1}, G/G_{r-1}) \sim (K_1/K_0, K_2/K_1, \ldots, K_t/K_{t-1}, M/K, G/M).$$

Similarly $(**)$ gives,

$$(H_1/H_0, H_2/H_1, \ldots, H_{s-2}/H_{s-1}, G/H_{s-1}) \sim (K_1/K_0, K_2/K_1, \ldots, K_t/K_{t-1}, N/K, G/N).$$

The right hand sides of the above equations are identical except for the last two elements. And, by Lemma 3, we have $(M/K, G/M) \sim (N/K, G/N)$. $\square$