Problem Set 1 : Due Tuesday, January 17

See the course website for homework policy.

1. Let $\mathbb{F}_p$ denote the field $\mathbb{Z}/p\mathbb{Z}$.
   (a) Let M be a $k \times n$ matrix with entries in $\mathbb{F}_p$, with $k < n$. Suppose that M has rank k. Form a new matrix M' by adding an additional vector row vector v beneath M. Of the p^n possible choices for v, how many of them will make M' have rank $k+1$?
   $\mathrm{GL}_n(\mathbb{F}_p)$ is the group of invertible $n \times n$ matrices with entries in $\mathbb{F}_p$.
   (b) How many elements are in the group $\mathrm{GL}_n(\mathbb{F}_p)$?
   Let $G(k, \mathbb{F}_p^n)$ be the space of k-dimensional subspaces of $\mathbb{F}_p^n$.
   (c) Show that $\mathrm{GL}_n(\mathbb{F}_p)$ acts transitively on $G(k, \mathbb{F}_p^n)$.
   Let $L \in G(k, \mathbb{F}_p^n)$ be the span of the first k standard basis vectors.
   (d) Describe the stabilizer of L and compute the order of this stabilizer.
   (e) Compute the cardinality of $G(k, \mathbb{F}_p^n)$.

2. Let $O(3)$ be the group of three by three real orthogonal matrices, and $SO(3)$ the subgroup of three by three real orthogonal matrices of determinant 1. Give an isomorphism $O(3) \cong SO(3) \times C_2$.

3. Let A_5 be the group of even permutations in S_5. Let $\sigma = (123)$ and $\tau = (345)$
   (a) Show that $\sigma\tau$ and $\sigma\tau^2$ have order 5.
   (b) Show that there are no nontrivial group homomorphisms $\chi : A_5 \to \mathbb{C}^*$. You may assume that σ and τ generate A_5. (Hint: What can you say about $\chi(\sigma)$ and $\chi(\tau)$?)

4. Let G be a group and let A and B be subsgroups. Recall that $AB = \{ab : a \in A,\ b \in B\}$.
   (a) Give an example where AB is not a subgroup of G.
   (b) If A is a normal subgroup of G (but B need not be normal) show that AB is a subgroup of G.

5. Let G be a finite group and X a finite set on which G acts. For $g \in G$, let X^g be the number of elements of X fixed by g.
   (a) Show that the number of orbits of G acting on X is
   $$\frac{1}{|G|} \sum_{g \in G} |X^g|.$$
   (b) If G is a finite group acting transitively on X, with $|X| > 1$, show that some element of G acts on X with no fixed points.

(c) Give an example of a finite group G acting on a set X where all orbits have size greater than 1, but every element of G acts with some fixed point. (Hint: $G = C_2 \times C_2$ works.)

(d) Give an example of an infinite group G acting transitively on a set X of size > 1 such that every element of G has some fixed point on X.

6. Let G be a finite group. For $g \in G$, let $\mathrm{ord}(g)$ be the order of the element g. For g and $h \in G$, we define $g \equiv h$ if $h = g^k$ for some k with k relatively prime to $\mathrm{ord}(g)$.

   (a) Show that $\equiv$ is an equivalence relation.

   (b) For g and $h \in G$, define $g \approx h$ if there is some g' which is conjugate to g with $g' \equiv h$. Show that $\approx$ is an equivalence relation.

   (c) Let X be a finite set on which G acts, and suppose that $g \approx h$ for some g and h in G. For every integer i, show that the number of orbits of size i for g acting on X is the same as the number of orbits of size i for h acting on X.

   (d) Let G be a finite group and suppose that $g \not\approx h$ for some g and h in G. Construct a finite set X on which G acts so that the orbits of g on X have different sizes than the orbits of h.

7. Let G be a finite group. Let H be a subgroup of G with $n = [G : H]$.

   (a) Show that $[G : \bigcap_{x \in G} xHx^{-1}]$ divides $n!$.

   (b) Suppose that n is the smallest prime dividing $|G|$. Show that H is normal in G.