Problem Set 10 – Due Thursday, April 6

1. Let L be the splitting field of $x^4 - 2$ over $\mathbb{Q}$.
   (a) What is the Galois group of L over $\mathbb{Q}$?
   (b) Explicitly describe all subfields K of L such that $[K : \mathbb{Q}] = 4$.

2. Let $\theta = \sqrt{13 + 2\sqrt{13}}$.
   (a) Show that the polynomial $(x^2 - 13)^2 - 52$ splits in $\mathbb{Q}(\theta)$.
   (b) Show that $\mathbb{Q}(\theta)$ is Galois over $\mathbb{Q}$ with Galois group C_4.

3. Let α be a complex number. We define α to be an algebraic integer if it obeys a polynomial relation of the form $\alpha^n + c_1\alpha^{n-1} + \cdots + c_{n-1}\alpha + c_n = 0$ with the c_i integers.
   (a) Show that α is an algebraic integer if and only if the $\mathbb{Z}$ module spanned by 1, α, α^2, α^3, $\ldots$ is finitely generated over $\mathbb{Z}$.
   (b) Show that, if α and β are algebraic integers, so are $\alpha + \beta$ and $\alpha\beta$.
   (c) Show that, if α is an algebraic integer, then the minimal polynomial of α over $\mathbb{Q}$ is of the form $\alpha^n + c_1\alpha^{n-1} + \cdots + c_{n-1}\alpha + c_n = 0$ with the c_i integers. (Hint: Quote something from a previous problem set.)
   (d) Show that a rational number is an algebraic integer if and only if it is an integer.

4. Suppose that $p(x) \in \mathbb{Q}[x]$ is a polynomial of degree 5 with distinct roots α_1, α_2, α_3, α_4 and α_5. Suppose that $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)/\mathbb{Q}$ is Galois with Galois group S_5. Show that

$$\mathbb{Q}(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4) \cap \mathbb{Q}(2\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) = \mathbb{Q}.$$

5. This problem proves the Artin-Schrier Lemma, which is the characteristic p version of Kummer's Lemma. Let K be a field of characteristic $p > 0$, and let L/K be a Galois extension with Galois group C_p. This theorem says that $L \cong K[x]/(x^p - x - a)$ for some $a \in K$.

   Let σ generate C_p.
   (a) Let $\nu : L \to L$ be the map $x \mapsto \sigma(x) - x$. Show that there is some $y \in L$ such that $\nu(y) \neq 0$ but $\nu^2(y) = 0$. (Hint: What can we say about ν as a K-linear map?)
   Let y be as in the previous part and let $z = \nu(y)$. Set $x = y/z$.
   (b) Show that $z \in K$.
   (c) Show that $\sigma(x) = x + 1$.
   (d) Show that $x^p - x \in K$.
   Set $a = x^p - x$.
   (e) Show that $L = K(x)$ and the minimal polynomial of x is $x^p - x - a = 0$.