Problem Set 2 : Due Tuesday, January 24

See the course website for homework policy.

1. Let G be a group, and let H and K be subgroups of G. Let X be the set G/H. Show the action of K on X has a fixed point if and only if we have $K \subseteq gHg^{-1}$ for some $g \in G$.

2. Describe all possible actions of C_2 on the group C_{15}.

3. Let G be a group and g an element of G. Define an action of $\mathbb{Z}$ on G by $\phi(k)(h) = g^k h g^{-k}$, for $h \in G$ and $k \in \mathbb{Z}$. Show that $\mathbb{Z} \ltimes_\phi G \cong \mathbb{Z} \times G$.

4. Which of the following short exact sequences are semi-direct:
   (a) $0 \to A_5 \to S_5 \to C_2 \to 0$?
   (b) $0 \to C_2 \to C_6 \to C_3 \to 0$?
   (c) $0 \to C_3 \to C_9 \to C_3 \to 0$?
   (d) $0 \to \{1, i, -1, -i\} \to Q \to C_2 \to 0$, where Q is the eight element subgroup of the quaternions consisting of $\{\pm 1, \pm i, \pm j, \pm k\}$?

5. Let G be a subgroup of $GL_n(\mathbb{F}_p)$, with order p^k. Write V for the vector space $\mathbb{F}_p^n$.
   (a) Show that the action of G on V fixes a nonzero vector.
   (b) Show that, after changing bases in V, we can arrange for G to be contained in the subgroup

$$\begin{pmatrix} 1 & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ & & & \ddots & \\ 0 & * & * & \cdots & * \end{pmatrix}$$

   of $GL_n(\mathbb{F}_p)$.
   (c) Show that, after changing bases in V, we can arrange for G to be contained in the subgroup

$$\begin{pmatrix} 1 & * & * & \cdots & * \\ 0 & 1 & * & \cdots & * \\ 0 & 0 & 1 & \cdots & * \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Turn over for more group theory fun!

6. This question will walk you through the Jordan-Holder theorem for modules. It is easier than for groups! Let R be a ring. An R-module M is called **simple** if it has no submodules other than 0 and M. A **Jordan-Holder filtration** of M is a sequence of submodules $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_\ell = M$ such that M_{i+1}/M_i is simple.

   (a) Suppose that M has a Jordan-Holder filtration of length ℓ and P is a submodule of M. Show that P has a Jordan-Holder filtration of length $\leq \ell$.

   (b) Suppose that M has a Jordan-Holder filtration of length ℓ and P is a submodule of M. Show that M/P has a Jordan-Holder filtration of length $\leq \ell$.

   (c) Let M be a module and let A and B be simple submodules with $A \cap B = (0)$. Define $A + B = \{a + b : a \in A, b \in B\}$. Show that $A + B \cong A \oplus B$.

   (d) Let M have two Jordan-Holder filtrations $(M_0, M_1, \cdots, M_k)$ and $(N_0, N_1, \cdots, N_\ell)$. Show that $k = \ell$ and the M_{i+1}/M_i are a permutation of the N_{j+1}/N_j.

7. Let G be a **finite** group and let $\phi : G \to G$ be a homomorphism. We write ϕ^n for the n-fold composition of ϕ with itself. Let K^n be the kernel of ϕ^n and let I^n be the image of ϕ^n.

   (a) Show that $K^1 \subseteq K^2 \subseteq K^3 \subseteq \cdots$ and $I^1 \supseteq I^2 \supseteq I^3 \supseteq \cdots$. Show that there is some N such that $K^N = K^{N+1} = K^{N+2} = \cdots$ and $I^N = I^{N+1} = I^{N+2} = \cdots$.

   We define K^∞ and I^∞ to be I^N and K^N for N as in the above paragraph. The theme of this problem is that, if I^∞ is neither (e) nor G, we can decompose G into simpler groups.

   (b) Show that we have a short exact sequence $1 \to K^\infty \to G \to I^\infty \to 1$.

   (c) Show that $G \cong K^\infty \rtimes I^\infty$.

   (d) The homomorphism ϕ is called **normal** if $\phi(aba^{-1}) = a\phi(b)a^{-1}$ for all a and $b \in G$. Show that, if ϕ is normal, then $G \cong K^\infty \times I^\infty$.