Problem Set 3 : Due Tuesday, January 31

See the course website for homework policy.

1. If H is any group, we have a map $H \to \mathrm{Aut}(H)$ sending h to the automorphism $g \mapsto hgh^{-1}$. The group H is called ***complete*** this map is an isomorphism.
   (a) Show that, if H is complete, the center of H is trivial.
   (b) Show that, if H is complete and $1 \to H \to G \to Q \to 1$ is any short exact sequence, then $G \cong H \times Q$. (Hint: There is a natural map $G \to \mathrm{Aut}(H)$.)

2. Let G be a finite group and suppose that $\mathrm{Aut}(G)$ acts transitively on $G \setminus \{e\}$. In this problem, we will show that G is of the form C_p^k for some prime p and some k. We break this argument into three parts.
   (a) Show that $|G| = p^k$ for some prime p and some k.
   (b) Show that G is abelian.
   (c) Show that $G \cong C_p^k$.

3. Let G be a group which lies in two short exact sequences $1 \to N_1 \to G \to Q_1 \to 1$ and $1 \to N_2 \to G \to Q_2 \to 1$. Suppose the composition $N_1 \to G \to Q_2$ is an isomorphism.
   (a) Show that the composition $N_2 \to G \to Q_1$ is an isomorphism.
   (b) Show that both sequences are left split, so $G \cong N_1 \times G_1 \cong N_2 \times G_2$.

4. In Question 7 of the previous problem set, we introduced the notion of a **normal endomorphism** of a group – a homomorphism $\phi : G \to H$ satisfying $\phi(ghg^{-1}) = g\phi(h)g^{-1}$.
   (a) Let A_1 and A_2 be groups and $G = A_1 \times A_2$. Define $i : A_1 \to G$ by $i(a) = (a, e)$ and $p : G \to A_1$ by $p(a_1, a_2) = a_1$. Show that $i \circ p : G \to G$ is normal.
   (b) Let A_1, A_2, B_1 and B_2 be groups and let $G \cong A_1 \times A_2 \cong B_1 \times B_2$. Define $i : A_1 \to G$ and $p : G \to A_1$ as before, and define $j : B_1 \to G$ and $q : G \to B_1$ similarly. Show that $p \circ j \circ q \circ i : A_1 \to A_1$ is normal.

5. Let G be a group of order $2^n m$ with m odd, and suppose that G contains an element of order 2^n. Show that all elements of order 2 in G are conjugate.

6. Let $f : G \to H$ be a surjection of finite groups and let P be a p-Sylow of G. Show that $f(P)$ is a p-Sylow of H.

7. Let $H \subset G$ be finite groups and p a prime. Show that the number of p-Sylows of G is greater than or equal to that of H.