Problem Set 4 : Due Thursday, February 9

See the course website for homework policy.

1. Let B be the group generated by f, g and h, subject to the relations
$$f^2 = g^2 = h^2 = 1 \quad fgf = gfg \quad fh = hf \quad ghgh = hghg.$$
Compute the abelianization of B.

2. Let G be a group such that $\mathrm{Aut}(G)$ is nilpotent. Show that G is nilpotent.

3. (a) Let G be a group and let S_1 and S_2 be normal solvable subgroups. Show that $S_1 S_2$ is a normal solvable subgroup.
   (b) Let G be a finite group. Show that G contains a normal solvable subgroup S such that, if S' is any normal solvable subgroup of G, then $S' \subseteq S$.

   Remark: The analogous results hold for normal nilpotent subgroups, but the analogue of part (a) is a bit harder.

4. In this problem you will prove Burnside's basis theorem: If G is a p-group, and $g_1, g_2, \ldots, g_m$ are elements of G generating G^{ab}/pG^{ab}, then $g_1, \ldots, g_m$ generate G.
   (a) Let G be a p-group and $H \subsetneq G$ a proper subgroup. Show that there is some g in G, but **not** in H, such that $gHg^{-1} = H$. (Hint: I know two approaches. The first starts : "Case 1, suppose $Z(G) \not\subseteq H$ ..." and the second starts "Consider the set ..., whose cardinality is size p^j, and the action of the p-group ... on it ....")
   (b) Let M be a maximal subgroup of G, meaning that $M \subsetneq G$ but there is no subgroup M' with $M \subsetneq M' \subsetneq G$. Show that M is normal in G.
   (c) With notation as in the previous part, show that $G/M \cong C_p$.
   (d) Let $g_1, g_2, \ldots, g_m$ be elements of G which do **not** generate G. Show that there is a nontrivial homomorphism $G \to C_p$ whose kernel contains all of the g_i.
   (e) Prove Burnside's basis theorem: If G is a p-group, and $g_1, g_2, \ldots, g_m$ are elements of G generating G^{ab}/pG^{ab}, then $g_1, \ldots, g_m$ generate G.

   Remark: Part (a) is true for all nilpotent groups (finite and infinite), not just p-groups. If you establish part (a) in that level of generality, and then follow the logic of the rest of the problem, you will show that, if G is a nilpotent group and $g_1, g_2, \ldots, g_m$ generate G^{ab}, then they generate G.

Turn over for more group theory fun!

5. Let p be an odd prime. (The big picture is the same for $p = 2$, but the details are messier.) The goal of this problem is to show that there are many non-isomorphic groups of order p^N. Specifically, we will show there are at least $p^{\frac{2}{27}N^3 - O(N^2)}$ non-isomorphic groups of this order.

   Let V be an $\mathbb{F}_p$-vector space of dimension j.

   (a) Define a group operation on the set $V \times \bigwedge^2 V$ by $(v_1, \omega_1) \cdot (v_2, \omega_2) = (v_1 + v_2, \omega_1 + \omega_2 + v_1 \wedge v_2)$. Check that this is a group operation. We will call this group Γ.

   (b) Let W be any subspace of $\bigwedge^2 V$. Show that W is a normal subgroup of Γ.

   Our many different groups will be of the form Γ/W where $j + \binom{j}{2} - \dim W = N$. We write $G(d, \bigwedge^2 V)$ for the set of d-dimensional subspaces of $\bigwedge^2 V$.

   (c) Let G be a group of the form Γ/W. Show that $[G, G] \cong \bigwedge^2 V/W$ (this uses that p is odd) and $G^{ab} \cong V$.

   (d) Let G_1 and G_2 be **isomorphic** groups of the form Γ/W_1 and Γ/W_2 with $\dim W_1 = \dim W_2 = d$. Show that there is an element $g \in \mathrm{GL}(V)$ taking W_1 to W_2 under the natural action of $\mathrm{GL}(V)$ on $G(d, \bigwedge^2 V)$. (Hint: An isomorphism $G_1 \to G_2$ induces an isomorphism $G_1^{ab} \to G_2^{ab}$.)

   At this point, we know that the number of non-isomorphic groups of order p^N is at least the number of orbits of $GL(V)$ on $G(\binom{j}{2} - k, \bigwedge^2 V)$ where $j + k = N$. The remainder of the problem is to optimize this quantity:

   (e) Let U be an $\mathbb{F}_p$ vector space of dimension b. In this problem, we will show that the number of a-dimensional subspaces of U is at least $p^{a(b-a)}$. Choose a basis $e_1, e_2, \ldots, e_b$ of U and let $X_0 = \mathrm{Span}(e_1, e_2, \ldots, e_a)$. Let H be the subgroup $\begin{pmatrix} \mathrm{Id}_a & 0 \\ * & \mathrm{Id}_{b-a} \end{pmatrix}$. Show that the orbit of H through the point X_0 of $G(a, U)$ has order $p^{a(b-a)}$.

   (f) Let N be divisible by 3. Show that the number of non-isomorphic groups of order p^N is at least p^M where $M = \frac{N}{3}\left(\binom{2N/3}{2} - \frac{N}{3}\right) - \frac{N^2}{9} = \frac{2}{27}N^3 - O(N^2)$.