Problem Set 5 : Due Thursday, February 16

See the course website for homework policy.

1. Let G be a group. Recall that we defined the commutator subgroup, $[G, G]$, to be the subgroup *generated by* all elements of the form $ghg^{-1}h^{-1}$. In this problem, we will construct a group where not all elements of G are of the form $ghg^{-1}h^{-1}$.

   Let k be a field. Let G be the group of 7×7 matrices with entries in k of the form

$$\begin{pmatrix} 1 & 0 & 0 & u_1 & A_{11} & A_{12} & A_{13} \\ 0 & 1 & 0 & u_2 & A_{21} & A_{22} & A_{23} \\ 0 & 0 & 1 & u_3 & A_{31} & A_{32} & A_{33} \\ 0 & 0 & 0 & 1 & v_1 & v_2 & v_3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

   We will abbreviate this matrix as $g(u, v, A)$ where u and v are length 3 column vectors, and A is a 3×3 matrix.

   (a) Verify that $g(u, v, A)^{-1} = g(-u, -v, -A + uv^T)$. (Note that uv^T is a 3×3 matrix.)

   (b) Show that $g(u, v, A)g(u', v', A')g(u, v, A)^{-1}g(u', v', A')^{-1}$ is of the form $g(0, 0, B)$, and give a formula for B in terms of u, v, A, u', v', A'.

   (c) Show that the matrix B in part (b) has rank ≤ 2.

   (d) Show that, for any 3×3 matrix C, the element $g(0, 0, C)$ is in the group $[G, G]$.

2. In this problem, we show that every finite nilpotent group is a product of p-groups.

   (a) Let p be prime and let $1 \to P \to G_1 \to G_2 \to 1$ be a short exact sequence where P is a p-group. Let P_r be a p-Sylow of G_r. Show that, if $P_2 \lhd G_2$, then $P_1 \lhd G_1$.

   (b) Let p be prime and let $1 \to Z \to G_1 \to G_2 \to 1$ be a short exact sequence where Z is central and $p \not| \#(Z)$. Let P_r be a p-Sylow of G_r. Show that, if $P_2 \lhd G_2$, then $P_1 \lhd G_1$.

   (c) Let G be a finite nilpotent group and let P be a p-Sylow of G. Show that $P \lhd G$.

   (d) Let $|G|$ have prime factorization $\prod p_i^{a_i}$ and let P_i be a p_i-Sylow. Show that $G \cong \prod P_i$.

3. Let G be a group with $8 \cdot 7^m$ elements.

   (a) Show that the number of 7-Sylow subgroups of G is 1 or 8.

   (b) Show that, if S_8 is the symmetric group, and $\phi : G \to S_8$ is a group homomorphism, then the image of ϕ has ≤ 56 elements.

   (c) Show that G is solvable. You may use that all groups of order < 60 are solvable.